Journal of Statistical Physics, Vol. 78, Nos. 5/6, 1995

Discrete Symmetry Groups of Vertex Models in Statistical Mechanics

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Received May 2, 1994; final September 8, 1994

We analyze discrete symmetry groups of vertex models in lattice statistical mechanics represented as groups of birational transformations. They can be seen as generated by involutions corresponding respectively to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and an (involutive) permutation of the entries of the matrix. We show that the analysis of the factorizations of the iterations of these transformations is a precious tool in the study of lattice models in statistical mechanics. This approach enables one to analyze two-dimensional q^4-state vertex models as simply as three-dimensional vertex models, or higher-dimensional vertex models. Various examples of birational symmetries of vertex models are analyzed. A particular emphasis is devoted to a three-dimensional vertex model, the 64-state cubic vertex model, which exhibits a polynomial growth of the complexity of the calculations. A subcase of this general model is seen to yield integrable recursion relations. We also concentrate on a specific two-dimensional vertex model to see how the generic exponential growth of the calculations reduces to a polynomial growth when the model becomes Yang–Baxter integrable. It is also underlined that a polynomial growth of the complexity of these iterations can occur even for transformations yielding algebraic surfaces, or higher-dimensional algebraic varieties.

KEY WORDS: Birational transformations; vertex models; inversion trick; discrete dynamical systems; nonlinear recursion relations; iterations; integrable mappings; elliptic curves; Abelian surfaces; Jacobian of algebraic curves; automorphisms of algebraic varieties; complexity of iterations; polynomial growth.

1. INTRODUCTION

In previous papers[1–3] we have analyzed birational representations of discrete groups generated by involutions having their origin in the theory of

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exactly solvable models in lattice statistical mechanics.[4-11] These involutions correspond respectively to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and an (involutive) permutation of the entries of the matrix.

The analysis of birational representations of discrete symmetry groups of the parameter space of vertex models has been a powerful tool in lattice statistical mechanics.[4-11] The methods developed in these papers are of two different types: a systematic search of algebraic expressions invariant under these discrete groups of symmetries and a visualization of (two-dimensional projections of) the orbits of these groups of symmetries. When considering three-dimensional (or higher-dimensional) vertex models the number of parameters of these models quickly becomes large (64 homogeneous parameters,...). It becomes difficult to get an exhaustive list of algebraic invariants of these groups, or equivalently the equations defining the algebraic variety corresponding to these orbits. From the point of view of effective algebraic geometry it is hard, because of the large number of the variables, to characterize the nature of the algebraic variety. On the other hand, the visualization of the orbits often provides a very efficient way to describe these orbits when they are fractal-like sets of points[5,6] or, on the contrary, curves foliating the whole parameter space or even surfaces, the action of one of the infinite-order transformations being like a shift on a torus (see Fig. 1 in the following). However, it is much harder to get some hint on the very nature of these orbits when they look fuzzy (see, for instance, Fig. 2 in the following). For such cases these methods are no longer appropriate. There is a need for a complementary approach. The analysis of the factorization properties of these discrete groups of symmetries[1-3] performed here *provides such a complementary approach.* In this framework a quite general three-dimensional vertex odel (64 homogeneous parameters) will surprisingly be seen as remarkably interesting.

In refs. 1–3, permutations of the entries corresponding to *permutations of two entries* were analyzed. For these permutations, it has been shown that the iteration of the associated birational transformations presents some remarkable factorization properties.[1,2] These factorization properties explain why the complexity of these iterations, instead of having the exponential growth one expects at first sight, may have a *polynomial growth.*[12-14] It has also been shown that the polynomial factors occurring in these factorizations *can satisfy nonlinear recursion relations* and that some of these recursions are actually *integrable* yielding *elliptic curves.*[1-3] These papers have tried, on simple examples of permutations, to shed some light on the relation between various structures and properties, such as the factorization of the iterations, the polynomial growth of the complexity,[1,2,12] and the integrability of the mappings, as well as the nature of the various algebraic varieties preserved by these mappings.

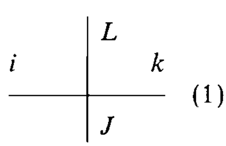
The structures, concepts, properties, and results that emerged in these studies will be used here[2] *for lattice models of statistical mechanics: vertex models and also (edge) spin models.* The mappings that one considers here are birational representations of symmetries acting in the parameter space of vertex models in two, three, or even *arbitrary* dimensions.

We will first concentrate on a specific two-dimensional q^4-vertex model, with emphasis on $q = 3$, to see *how the generic exponential growth of the calculations reduces to a polynomial growth when the model becomes Yang–Baxter integrable.* Special attention will also be devoted to a three-dimensional vertex model, the 64-state cubic vertex model, which exhibits a polynomial growth of the complexity of the calculations. It will be shown that a *polynomial growth* of the complexity of these iterations can occur not only for transformations yielding algebraic curves, but also for *transformations yielding algebraic surfaces, or higher-dimensional algebraic varieties.*[3] A particular subcase of this three-dimensional vertex model, for which one of the infinite-order discrete symmetry mappings is integrable, will be seen to yield remarkable recursion relations (see Section 4.2.2) providing a new exact result on a three-dimensional vertex model.

This factorization analysis provides a new approach for q^d-vertex models with *arbitrary* number q of colors and *arbitrary* lattice dimension d. In this new approach for vertex models the *occurrence of polynomial growth* becomes a necessary condition for selecting interesting vertex models.

2. GENERAL FRAMEWORK AND NOTATIONS

Let us consider the more general vertex model where one direction, denoted direction 1, is singled out. Pictorially this can be interpreted as follows:



(2.1)

where i and k (corresponding to direction 1) can take q values, while J and L take m values.

[2] Except for a few reminders in Sections 3.1 and 4.2.1 and Appendix C of some results of the previously mentioned series of papers, all the results presented here are new.

[3] This happens at least when one can associate a Jacobian variety to these birational transformations, as will be seen in the following.

One can define a "partial" transposition on direction 1 denoted t_1. The action of t_1 on the R-matrix is given by[10]

$$(t_1 R)^{iJ}_{kL} = R^{kJ}_{iL} \tag{2.2}$$

The R-matrix is a $(qm)\times(qm)$ matrix which can be seen as q^2 blocks which are $m\times m$ matrices:

$$R = \begin{pmatrix} A[1,1] & A[1,2] & A[1,3] & \cdots & A[1,q] \\ A[2,1] & A[2,2] & A[2,3] & \cdots & A[2,q] \\ A[3,1] & A[3,2] & A[3,3] & \cdots & A[3,q] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A[q,1] & A[q,2] & A[q,3] & \cdots & A[q,q] \end{pmatrix} \tag{2.3}$$

where $A[1,1], A[1,2],..., A[q,q]$ are $m\times m$ matrices. With these notations the partial transposition t_1 amounts to a permutation of matrices $A[\alpha,\beta]$ and $A[\beta,\alpha]$.

We use the same notations as in refs. 1–3, that is, we introduce the following transformations, the matrix inverse $\hat{I}$ and the homogeneous matrix inverse I:

$$\hat{I}:\quad R \to R^{-1} \tag{2.4}$$

$$I:\quad R \to \det(R)\cdot R^{-1} \tag{2.5}$$

The homogeneous inverse I is a polynomial transformation on each of the entries of matrix R, which associates with each entry its corresponding cofactor.

The two transformations t_1 and $\hat{I}$ are *involutions* and $I^2 = (\det(R))^{qm-2}\cdot \mathcal{I}d$ where $\mathcal{I}d$ denotes the identity transformation.

We also introduce the (generically infinite-order) transformations

$$K = t_1 \cdot I$$

and (2.6)

$$\hat{K} = t_1 \cdot \hat{I}$$

Transformation $\hat{K}$ is clearly a *birational transformation* on the entries of matrix R, since its inverse transformation is $\hat{I}\cdot t_1$, which is obviously a rational transformation. K is a *homogeneous polynomial transformation* on the entries of matrix R.

3. TWO-DIMENSIONAL VERTEX MODELS

3.1. Iterations Associated with the Sixteen-Vertex Model

The *16-vertex model* corresponds to the vertex of (2.1) and the R-matrix (2.3) with $q = m = 2$. In this case of 4×4 matrices, permutation t_1 has already been introduced in the framework of the symmetries of the 16-vertex model.[10] Namely, t_1 amounts to a permutation of two 2×2 (off-diagonal) submatrices of the 4×4 matrix R.

Remarkably, the symmetry group generated by the matrix inverse $\hat{I}$ and transformation t_1, or the infinite generator $\hat{K} = t_1 \cdot \hat{I}$, has been shown to yield *algebraic elliptic curves*[10] in $\mathbb{C}P_{15}$.

Let us consider a 4×4 matrix M_0 and the successive matrices obtained by iteration of transformation $K = t_1 \cdot I$. Remarkably, all the entries of the successive matrices obtained iterating K on M_0 do factorize common polynomials. This enables us to introduce at each step reduced matrices, denoted M_n. Moreover, the determinants of these M_n also factorize. More precisely, similarly to factorizations described in refs. 1 and 2, one has the following factorizations for the iterations of K,[1]

$$M_1 = K(M_0), \qquad M_2 = K(M_1), \qquad F_1 = \det(M_0),$$

$$F_2 = \det(M_1), \qquad F_3 = \frac{\det(M_2)}{F_1^3}, \qquad M_3 = \frac{K(M_2)}{F_1^2}, \ldots$$

and for arbitrary n

$$M_{n+2} = \frac{K(M_{n+1})}{F_n^2}, \qquad F_{n+2} = \frac{\det(M_{n+1})}{F_n^3} \tag{3.1}$$

One also has the following relation:

$$\hat{K}(M_{n+2}) = \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{F_{n+1} F_{n+3}} \tag{3.2}$$

One can also introduce a *right action* of K on matrices M_n or on any homogeneous polynomial expression of their entries (such as the F_n, for instance): the entries of matrix M_0 are replaced by the entries of $K(M_0)$. Amazingly, the right action of K on the F_n and on the matrices M_n factorizes F_1 *and only* F_1,[1]

$$(M_n)_K = M_{n+1} \cdot F_1^{\nu_n}, \qquad (F_n)_K = F_{n+1} \cdot F_1^{\mu_n} \tag{3.3}$$

Denoting α_n the degree of the determinant of the matrix M_n and β_n the degree of the polynomial F_n, one immediately gets from equations (3.1) and (3.2) the following linear relations (with integer coefficients):

$$\alpha_{n+1} = 3\beta_n + \beta_{n+2}, \qquad 3\alpha_{n+1} = \alpha_{n+2} + 8\beta_n$$
$$\alpha_{n+2} + \alpha_{n+3} = 4(\beta_{n+1} + \beta_{n+3}), \qquad 3\beta_n = \beta_{n+1} + 4\mu_n, \qquad 3\alpha_n = \alpha_{n+1} + 16\nu_n \tag{3.4}$$

yielding the generating functions

$$x\alpha(x) = (3x^2+1)\cdot\beta(x), \qquad (3x-1)\cdot\alpha(x) + 4 = 8x^2\cdot\beta(x)$$
$$(1+x)\cdot\alpha(x) = 4(1+x^2)\cdot\beta(x) + 4 \tag{3.5}$$

From these factorizations, one can easily deduce linear recursions on the series α_n, β_n, μ_n, and ν_n and then the following expressions for their generating functions:

$$\alpha(x) = \frac{4(1+3x^2)}{(1-x)^3}, \qquad \beta(x) = \frac{4x}{(1-x)^3},$$
$$\mu(x) = \frac{x^2(3-x)}{(1-x)^3}, \qquad \nu(x) = \frac{2x^2}{(1-x)^3} \tag{3.6}$$

The expressions of the degrees and exponents α_n, β_n, μ_n, and ν_n, respectively, read

$$\alpha_n = 4(2n^2+1), \qquad \beta_n = 2n(n+1), \qquad \mu_n = n^2 - 1, \qquad \nu_n = n(n-1) \tag{3.7}$$

Let us also mention that, for a given initial matrix M_0, the successive iterates of M_0 under transformation K^2 move in a *three-dimensional affine projective space*:

$$K^{2n}(M_0) = a_0^{(n)}\cdot M_0 + a_1^{(n)}\cdot M_2 + a_2^{(n)}\cdot M_4 + a_3^{(n)}\cdot M_6 \tag{3.8}$$
$$K^{2n+1}(M_0) = b_0^{(n)}\cdot M_1 + b_1^{(n)}\cdot M_3 + b_2^{(n)}\cdot M_5 + b_3^{(n)}\cdot M_7 \tag{3.9}$$

In terms of these homogeneous variables a_0^n, a_1^n, a_2^n, a_3^n (or b_0^n, b_1^n, b_2^n, b_3^n) the transformation K is represented as a *cubic* (birational) homogeneous transformation:

$$a_i^{(n)} \to b_i^{(n)} = \sum_{N_0+N_1+N_2+N_3=3} B_i(M_0; N_0, N_1, N_2, N_3)$$
$$\cdot (a_0^{(n)})^{N_0}\cdot(a_1^{(n)})^{N_1}\cdot(a_2^{(n)})^{N_2}\cdot(a_3^{(n)})^{N_3} \qquad \text{with} \quad i = 0, 1, 2, 3 \tag{3.10}$$

(the N_i are positive integers), and similarly

$$b_i^{(n)} \to a_i^{(n+1)} = \sum_{N_0+N_1+N_2+N_3=3} A_i(M_0; N_0, N_1, N_2, N_3) \cdot (b_0^{(n)})^{N_0} \cdot (b_1^{(n)})^{N_1} \cdot (b_2^{(n)})^{N_2} \cdot (b_3^{(n)})^{N_3} \quad \text{with} \quad i=0, 1, 2, 3 \tag{3.11}$$

Considering the points in $\mathbb{C}P_{15}$ associated with the successive 4×4 matrices corresponding to the iteration of M_0 under the transformation K (instead of K^2), one thus gets sets of points (lying on elliptic curves) which belong to *two* three-dimensional affine subspaces of $\mathbb{C}P_{15}$, *which also depend on the initial matrix* M_0 in a quite involved way.

Amazingly, the F_n satisfy *a whole hierarchy of recursion relations*,[1] such as

$$\frac{F_n F_{n+3}^2 - F_{n+4} F_{n+1}^2}{F_{n-1} F_{n+3} F_{n+4} - F_n F_{n+1} F_{n+5}} = \frac{F_{n-1} F_{n+2}^2 - F_{n+3} F_n^2}{F_{n-2} F_{n+2} F_{n+3} - F_{n-1} F_n F_{n+4}} \tag{3.12}$$

Let us recall that this very recursion is integrable, yielding algebraic elliptic curves.[1]

Let us remark that, for the 16-vertex model, the two directions are equivalent. Therefore, in this factorization analysis, one can replace the transposition t_1 by the transposition on direction (2), denoted t_2. It amounts to a relabeling of the rows and columns of the R-matrix. In fact, the product $t_1 \cdot t_2$ is nothing but the "total" transposition of matrix R, and thus commutes with I.

Let us now consider new examples of vertex models.

3.2. q^4-State Two-Dimensional Vertex Models

Let us consider a generalization of the 16-vertex model for an arbitrary number of spin values. It corresponds to $m = q$ for model (2.1). The matrix (2.3) is now a $q^2 \times q^2$ matrix.

Similarly to the factorizations described in (3.1), one has the following factorizations for the iterations of K acting on M_0, a $q^2 \times q^2$ matrix:

$$M_1 = K(M_0), \qquad M_2 = K(M_1), \qquad F_1 = \det(M_0),$$

$$F_2 = \det(M_1), \qquad F_3 = \frac{\det(M_2)}{F_1^{q^2-1}}, \ldots$$

and for arbitrary n

$$M_{n+2} = \frac{K(M_{n+1})}{F_n^{q^2-2}}, \qquad F_{n+2} = \frac{\det(M_{n+1})}{F_n^{q^2-1}} \tag{3.13}$$

One recovers relation (3.2), *independently of q*:

$$\hat{K}(M_{n+2}) = \frac{K(M_{n+2})}{\det(M_{n+2})} = \frac{M_{n+3}}{F_{n+1}F_{n+3}} \tag{3.14}$$

Moreover, the action of the transformation K again yields the factorization of F_1 and only F_1, enabling us to define the exponents μ_n and ν_n.

It is clear that these factorizations are straightforward generalizations of the one described in Section 3.1. From these factorizations, one can easily get linear recursion relations for the exponents α_n, β_n, μ_n, and ν_n:

$$\begin{gathered}
\alpha_{n+3} + \alpha_{n+2} = q^2 \cdot (\beta_{n+1} + \beta_{n+3}), \qquad \alpha_{n+1} = (q^2 - 1) \cdot \beta_n + \beta_{n+2} \\
(q^2 - 1) \cdot \alpha_{n+1} = \alpha_{n+2} + (q^2 - 2) \cdot \beta_n \\
(q^2 - 1) \cdot \beta_n = \beta_{n+1} + q^2 \cdot \mu_n, \qquad (q^2 - 1) \cdot \alpha_n = \alpha_{n+1} + q^2 \cdot \nu_n
\end{gathered} \tag{3.15}$$

One deduces the relations in their generating functions:

$$\begin{gathered}
x \cdot \alpha(x) = [1 + (q^2 - 1)\, x^2] \cdot \beta(x), \qquad (1 + x) \cdot \alpha(x) = q^2(1 + x^2) \cdot \beta(x) + q^2 \\
q^2(q^2 - 2) \cdot x^2\beta(x) = [(q^2 - 1)\, x - 1] \cdot \alpha(x) + q^2 \\
q^2 \cdot x\mu(x) - q^2 x = [(q^2 - 1)\, x - 1] \cdot \beta(x) \\
q^4 \cdot x\nu(x) - q^2 = [(q^2 - 1)\, x - 1] \cdot \alpha(x)
\end{gathered} \tag{3.16}$$

and the following expressions for these generating functions:

$$\begin{aligned}
\alpha(x) &= \frac{q^2 \cdot [1 + (q^2 - 1)\, x^2]}{(1 - x) \cdot [1 - (q^2 - 2)\, x + x^2]} \\
\beta(x) &= \frac{q^2 x}{(1 - x) \cdot [1 - (q^2 - 2)\, x + x^2]} \\
\mu(x) &= \frac{x^2 \cdot [(q^2 - 1) - x]}{(1 - x) \cdot [1 - (q^2 - 2)\, x + x^2]} \\
\nu(x) &= \frac{(q^2 - 2) \cdot x^2}{(1 - x) \cdot [1 - (q^2 - 2)\, x + x^2]}
\end{aligned} \tag{3.17}$$

The expressions of the exponents α_n, β_n, μ_n, and ν_n clearly have (generically) an *exponential growth in terms of n* when q is different from 2. This suggests that the q^4-state vertex models are not generically "good candidates" for integrability when the number of colors q is no longer 2.

Let us recall that a *polynomial growth* of the calculations corresponds to cases where the roots of the denominators of the generating functions

$\alpha(x)$, $\beta(x)$,... are *Nth roots of unity*. On the explicit expressions (3.17) one sees that such a situation can only occur when q^2 is a *Tutte–Beraha number*,[15,16]

$$q^2 = 2 + t + \frac{1}{t}, \qquad \text{with} \quad t^N = 1 \quad \text{for some integer } N \tag{3.18}$$

A polynomial growth behavior cannot *generically* occur when q is an *integer* different from 2 (or 0...). In general, one does not expect the birational transformations defined in refs. 1–3 from permutations of entries of a $q \times q$ matrix to be integrable mappings.[4] It is, however, important to recall that integrable cases are not ruled out even when q is an integer different from 2.

3.3. From Exponential Growth to Integrability

It is known that there do exist "Yang–Baxter-integrable" subcases of the generic 9×9 matrix (and more generally of the generic $q^2 \times q^2$ matrix[18]): how is it possible for such integrable cases to survive in such a "hostile framework" (exponential growth of the complexity...)? Do these restrictions on the 9×9 matrices change the (generic) exponential growth of the calculations into a polynomial one?

For heuristic reasons let us, for example, consider a simple pattern for a 9×9 matrix corresponding to a vertex model introduced by Stroganov,[19]

$$R_{\text{Strog}} = \begin{pmatrix} 1 & 0 & 0 & 0 & b & 0 & 0 & 0 & b \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ b & 0 & 0 & 0 & b & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.19}$$

[4] It is, however, worth recalling the example of the q-state standard scalar Potts model, for which, using the Lieb–Temperley algebra, one can give a matrix representation in terms of matrices of sizes *independent of* q, q becoming a parameter in the entries of these matrices.[17] Therefore one can also imagine being able to define the birational transformations K for *noninteger values* of q.

This model is known to possess two "Yang–Baxter-integrable" subcases[19]:

$$c = 1 - b \qquad \text{and} \qquad c = \frac{1-b}{1+b} \tag{3.20}$$

When restricted to one of the two integrability subcases (3.20), the partition function of this model can easily be calculated using the *inversion trick.*[19] We consider this example because it is simple (only two parameters and a single one in the integrable subcases) and yields a *rational* parametrization of the integrable subcases of the model.[5] We restrict consideration to the first integrable subcase (3.20): $c = 1 - b$.

Let us first remark that, in this (rational) subcase, all the birational transformation symmetries we consider are just *homographic* transformations. For instance, the transformations t_1, I, and K^n read

$$\begin{gathered} t_1\colon \quad b \to c = 1 - b \\ I\colon \quad b \to \frac{-b}{1+b} \\ K\colon \quad b \to \frac{2b+1}{b+1}, \qquad K^n\colon \quad b \to \frac{N_n(b)}{D_n(b)} \end{gathered} \tag{3.21}$$

where the numerators and denominators of the first successive homographies K^n respectively read

$$\begin{gathered} N_1 = 2b+1, \qquad N_2 = 5b+3, \qquad N_3 = 13b+8, \qquad N_4 = 34b+21, \ldots \\ D_1 = b+1, \quad D_2 = 3b+2, \quad D_3 = 8b+5, \quad D_4 = 21b+13, \quad D_5 = 55b+34, \ldots \end{gathered} \tag{3.22}$$

These successive polynomials can be shown to satisfy the following recurrences:

$$N_{n+1} = 2N_n + D_n, \qquad D_{n+1} = N_n + D_n \tag{3.23}$$

The F_n and the entries of the successive matrices M_n, previously defined for generic 9×9 matrices [see (3.13) with $q = 3$], do factorize:

$$F_1 = -N_1 \cdot (b-1)^8, \quad F_2 = -N_2 \cdot (b-1)^{63} \cdot b^8, \quad F_3 = -N_3 \cdot (b-1)^{440} \cdot b^{63}$$

[5] Of course the reader can, as an exercise, replace this model and this form of the 9×9 matrix (3.19) by other "Yang–Baxter-integrable" 9×9 matrix patterns, for instance, the solvable q^4-state models introduced in ref. 18 among many other possibilities.

and for arbitrary n

$$F_n = -N_n \cdot (b-1)^{r_n} \cdot b^{r_{n-1}} \tag{3.24}$$

where the r_n are the coefficients of the rational function

$$r(x) = 1 + r_1 \cdot x + r_2 \cdot x^2 + r_3 \cdot x^3 + \cdots = \frac{1 + 7x^2 - x^3}{(1-x)(1-7x+x^2)} \tag{3.25}$$

All the entries of the M_n factorize the same polynomial, which enables us to introduce new matrices M_n^{int} whose entries are *polynomial* expressions in b:

$$M_1 = -M_1^{\text{int}} \cdot (b-1)^7, \qquad M_2 = -M_2^{\text{Strog}} \cdot (b-1)^{56} \cdot b^7,$$
$$M_3 = -M_3^{\text{int}} \cdot (b-1)^{392} \cdot b^{56}, \ldots$$

and for arbitrary n

$$M_n = -M_n^{\text{int}} \cdot (b-1)^{s_n} \cdot b^{s_{n-1}} \tag{3.26}$$

where the s_n are the coefficients of the rational function

$$s(x) = 1 + s_1 \cdot x + s_2 \cdot x^2 + s_3 \cdot x^3 + \cdots = \frac{1 - x + 8x^2 - x^3}{(1-x)(1-7x+x^2)} \tag{3.27}$$

Let us see how these new factorizations (3.24) and (3.26) *are actually compatible with the generic ones* (3.13).

The new (highly) factorized matrices M_n^{int} have a remarkably simple form in terms of the N_n and D_n:

$$M_n^{\text{int}} = \begin{pmatrix} D_n & 0 & 0 & 0 & N_n & 0 & 0 & 0 & N_n \\ 0 & 0 & 0 & -N_{n-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -N_{n-1} & 0 & 0 \\ 0 & -N_{n-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ N_n & 0 & 0 & 0 & D_n & 0 & 0 & 0 & N_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -N_{n-1} & 0 \\ 0 & 0 & -N_{n-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -N_{n-1} & 0 & 0 & 0 \\ N_n & 0 & 0 & 0 & N_n & 0 & 0 & 0 & D_n \end{pmatrix} \tag{3.28}$$

As it should, this matrix (up to the normalization of $R[1, 1]$) has exactly the same form as (3.19) where b has been changed into $K^n(b)$ [taking relation (3.23) into account]. The determinant of $M_n^{\rm int}$ can easily be calculated:

$$\det(M_n^{\rm int}) = -N_{n-1}^6 \cdot (2N_n + D_n) \cdot (N_n - D_n)^2 \tag{3.29}$$

Recalling recursions (3.23), we have that this expression reads

$$\det(M_n^{\rm int}) = -N_{n+1} \cdot N_{n-1}^8 \tag{3.30}$$

Recalling (3.24), (3.26), and (3.30) and factorization (3.13) for $q = 3$, one can write

$$\begin{aligned} F_{n+2} = \frac{\det(M_{n+1})}{F_n^8} &= \frac{\det(M_{n+1}^{\rm int}) \cdot (b-1)^{9s_{n+1}} \cdot b^{9s_n}}{[-N_n \cdot (b-1)^{r_n} \cdot b^{r_{n-1}}]^8} \\ &= -N_{n+2} \cdot (b-1)^{9s_{n+1} - 8r_n} \cdot b^{9s_n - 8r_{n-1}} \end{aligned} \tag{3.31}$$

This *compatibility between the factorizations for the generic* 9×9 *matrices (exponential growth) and the one for the Stroganov model* (3.19) *corresponds to the following relation on the* r_n, *the* s_n, *and the* β_n:

$$\beta_{n+2} = 9(s_{n+1} + s_n) - 8(r_n + r_{n-1}) + 1 \tag{3.32}$$

or, in terms of the associated generating functions $\beta(x)$, $r(x)$, and $s(x)$,

$$\begin{aligned} \beta(x) = 9x \cdot (1+x) \cdot s(x) - 8x^2 \cdot (1+x) \cdot r(x) \\ + \frac{1}{1-x} - 1 - 2x + 7x^2 + 8x^3 \end{aligned} \tag{3.33}$$

which is verified.

3.4. Stroganov's Model Outside the Yang–Baxter Integrability

When Stroganov's model is no longer restricted to the Yang–Baxter integrability conditions (3.20), the model, despite its simplicity [only two parameters, and a very simple form for the $q^2 \times q^2$ R-matrix (3.19),...] is not known to be integrable.

Let us examine the factorization properties *outside* the integrability conditions (3.20) [that is, in the whole (b, c) parameter space]. The first factorizations read

$$F_1 = -c^6 \cdot (2b+1) \cdot (b-1)^2, \qquad F_2 = c^{51} b^6 (b-1)^9 \cdot h_1 \cdot g_1^2, \ldots \tag{3.34}$$

with

$$g_1 = -1 - b + 2b^2 + c + bc, \qquad h_1 = -2 - 2b + 4b^2 - bc - c \tag{3.35}$$

All the entries of the matrix M_n factorize the same polynomial, which enables us to introduce new matrices M_n^{Strog} whose entries are *polynomial* expressions in b and c:

$$M_1 = M_1^{\mathrm{Strog}} \cdot (b-1) \cdot c^5, \qquad M_2 = -M_2^{\mathrm{Strog}} \cdot (b-1)^8 \cdot c^{45} \cdot g_1, \dots \tag{3.36}$$

Furthermore,

$$\begin{aligned} \det(M_1^{\mathrm{Strog}}) &= c^6 b^6 \cdot h_1 \cdot g_1^2, \\ \det(M_2^{\mathrm{Strog}}) &= c^6 b^6 (b-1)^6 \cdot (2b+1)^8 \cdot h_2 \cdot g_2^2, \dots \end{aligned} \tag{3.37}$$

with

$$\begin{aligned} g_2 &= 2 - 6b^2 + 4b^3 - c - cb + 2b^2 c - c^2 - bc^2 \\ h_2 &= 4 + 8b - 12b^2 - 16b^3 + 16b^4 - 2c - 3cb \\ &\quad + 3b^2 c + 2b^3 c - 2c^2 - 3bc^2 - b^2 c^2 \end{aligned} \tag{3.38}$$

The successive "reduced" matrices M_n^{Strog} also have a simple form slightly generalizing (3.28):

$$M_n^{\mathrm{Strog}} = \begin{pmatrix} A_n & 0 & 0 & 0 & B_n & 0 & 0 & 0 & B_n \\ 0 & 0 & 0 & -C_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -C_n & 0 & 0 \\ 0 & -C_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_n & 0 & 0 & 0 & A_n & 0 & 0 & 0 & B_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -C_n & 0 \\ 0 & 0 & -C_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -C_n & 0 & 0 & 0 \\ B_n & 0 & 0 & 0 & B_n & 0 & 0 & 0 & A_n \end{pmatrix} \tag{3.39}$$

There is no longer a relation between the A_n, B_n, and C_n [like $C_n = B_{n-1}$ for (3.28)]. With this particular form (3.39) the determinant of the "reduced" matrix factorizes, at least, as follows:

$$\det(M_n^{\mathrm{Strog}}) = -C_n^6 \cdot (2B_n + A_n) \cdot (B_n - A_n)^2 \tag{3.40}$$

The previous expressions g_1, h_1, g_2, h_2 simply read

$$g_1 = B_1 - A_1, \quad h_1 = 2B_1 + A_1, \quad g_2 = B_2 - A_2, \quad h_2 = 2B_2 + A_2 \tag{3.41}$$

Therefore one has a representation of K as a *birational transformation in* $\mathbb{C}P_2$:

$$(A_n, B_n, C_n) \rightarrow (A_{n+1}, B_{n+1}, C_{n+1})$$
$$= \left(\frac{C_n \cdot (B_n + A_n)}{H_n}, \frac{(B_n - A_n) \cdot (2B_n + A_n)}{H_n}, \frac{B_n \cdot C_n}{H_n} \right) \tag{3.42}$$

and a representation of K^{-1}:

$$(A_n, B_n, C_n) \rightarrow (A_{n-1}, B_{n-1}, C_{n-1})$$
$$= \left(\frac{B_n \cdot (C_n - A_n)}{L_n}, \frac{-B_n \cdot C_n}{L_n}, \frac{(A_n + C_n) \cdot (A_n - 2C_n)}{L_n} \right) \tag{3.43}$$

where H_n is the GCD polynomial of $C_n \cdot (B_n + A_n)$, $(B_n - A_n) \cdot (2B_n + A_n)$, and $B_n \cdot C_n$ and, similarly, L_n is the GCD polynomial of $B_n \cdot (C_n - A_n)$, $-B_n \cdot C_n$, $(A_n + C_n) \cdot (A_n - 2C_n)$, respectively. The transformation (3.42) can be written in a compact way:

$$M_{n+1}^{\mathrm{Strog}} = \frac{K(M_n^{\mathrm{Strog}})}{H_n} \tag{3.44}$$

The first A_n, B_n, C_n read

$$\begin{gathered} A_0 = 1, \qquad B_0 = b, \qquad C_0 = -c \\ A_1 = -c \cdot (b+1), \qquad B_1 = (b-1) \cdot (2b+1), \qquad C_1 = -bc \\ A_2 = -bc \cdot (2b^2 - b - c - bc - 1) \\ B_2 = (2b^2 - b + c + bc - 1) \cdot (4b^2 - 2b - c - bc - 2) \\ C_2 = -bc \cdot (b-1) \cdot (2b+1), \ldots \end{gathered} \tag{3.45}$$

Introducing the polynomials

$$X_n = B_{n-1} - A_{n-1}, \qquad Y_n = 2B_{n-1} + A_{n-1}, \qquad Z_n = B_{n-1} + A_{n-1} \tag{3.46}$$

we find that the polynomials A_{n+1}, B_{n+1}, C_{n+1} simply read

$$A_{n+1} = \frac{C_n}{H_n} \cdot Z_n, \qquad B_{n+1} = \frac{X_n}{H_n} \cdot Y_n, \qquad C_{n+1} = \frac{C_n}{H_n} \cdot B_n \tag{3.47}$$

The polynomials Y_n and Z_n do not factorize, with the polynomials X_n and the polynomials C_n (more precisely the C_{n-1}) *are divisible by* H_n. Defining the polynomial X_n^H by

$$X_n^H = \frac{X_n}{H_n} \tag{3.48}$$

one remarks that these X_n^H do not factorize. *Moreover, one also remarks that H_n is actually equal to Y_{n-2}*. The first expressions of the GCD H_n (or equivalently of the Y_{n-2}) read

$$\begin{aligned} &H_3 = Y_1 = 2b + 1, \qquad H_4 = Y_2 = 4b^2 - 2b - 2 - bc - c \\ &H_5 = Y_3 = 4 - 3bc - 12b^2 + 8b - 2c + 2b^3c \\ &\qquad\qquad + 3b^2c - b^2c^2 - 3bc^2 - 2c^2 + 16b^4 - 16b^3, ... \end{aligned} \tag{3.49}$$

In terms of these X_n^H, Y_n, and Z_n, the A_n, B_n, and C_n are

$$\begin{aligned} A_{n+1} &= X_{n-2}^H \cdot X_{n-3}^H \cdot X_{n-4}^H \cdots X_1^H \cdot B_0 \cdot C_0 \cdot Z_{n+1} \\ B_{n+1} &= X_{n+1}^H \cdot Y_{n+1} \\ C_{n+1} &= X_{n-1}^H \cdot X_{n-2}^H \cdot X_{n-3}^H \cdots X_1^H \cdot B_0 \cdot C_0 \cdot Y_n \end{aligned} \tag{3.50}$$

A more complete list of the successive expressions of X_n^H, Y_n, Z_n is given in Appendix A.

Representations (3.42) and (3.43) are nothing but the transformation K (or K^{-1}) represented as a birational transformation on (b, c). Let us, for instance, give the representation of I and t_1 as *birational* transformations on (b, c):

$$I: \quad (b, c) \rightarrow \left(\frac{-b}{1+b}, \frac{(1+2b)\cdot(1-b)}{(1+b)\cdot c}\right), \qquad t_1: \quad (b, c) \rightarrow (c, b) \tag{3.51}$$

Let us denote d_n the degree of polynomials A_n (or B_n or C_n) and $d(x)$ the associated generating function. The degrees of the X_n^H, Y_n, and Z_n, denoted d_n^X, d_n^Y, d_n^Z, respectively, satisfy

$$d_n^X = 1 + d_{n-2}, \qquad d_n^Y = d_n^Z = d_{n-1} \tag{3.52}$$

The relation $B_{n+1} = X_{n+1}^H \cdot Y_{n+1}$ yields

$$d_n = d_n^X + d_n^Y = 1 + d_{n-2} + d_{n-1}, \qquad d(x) \cdot (1 - x - x^2) - \frac{1}{1-x} = 0 \tag{3.53}$$

Thus $d(x)$ reads

$$\begin{aligned} d(x) &= 1 + d_1 \cdot x + d_2 \cdot x^2 + d_3 \cdot x^3 + \cdots = \frac{1}{(1-x)\cdot(1-x-x^2)} \\ &= 1 + 2x + 4x^2 + 7x^3 + 12x^4 + 20x^5 + 33x^6 \\ &\qquad + 54x^7 + 88x^8 + 143x^9 + 232x^{10} + \cdots \end{aligned} \tag{3.54}$$

The generating function $d(x)$ corresponds to an exponential growth of the complexity of the iterations like z^n where z is the largest root of $1+z-z^2$ ($z=1.618033989...$). This growth has to be compared with the exponential growth corresponding to the generic 9×9 matrices (see Section 3.2), for which one has an exponential growth like z^n where z is the largest root of $1-7z+z^2$ ($z=6.854101966...$).

This exponential growth for model (3.19) is confirmed by the fact that, seeking algebraic expressions of b and c $[P_1(b, c)/Q_1(b, c)]$ invariant under the transformation K [or the transformation I and t_1; see (3.51)], we have not found any such expressions up degree ten in b and c for $P_1(b, c)$ and $Q_1(b, c)$. One expects a quite "chaotic" behavior for the iterations of K outside the two integrability conditions (3.20).

3.5. Back to Integrability: The Inversion Trick

Since we claim that the various polynomials of the two variables (b, c) previously introduced (A_n, B_n, C_n, X_n^H, Y_n, Z_n,...) may be useful for a better understanding of Stroganov's model *outside* the Yang–Baxter integrability conditions (3.20), it is natural to look at these polynomials when restricted to the Yang–Baxter integrability conditions (3.20). For this purpose let us recall the (rational) well-suited parametrization of the model restricted to (3.20) (more precisely, $c=1-b$):

$$b=\frac{\omega x-\omega^{-1}}{1+x}, \qquad \text{where} \quad \omega=\frac{1+\sqrt{5}}{2} \tag{3.55}$$

and the expression of the partition function per site deduced from the *inversion trick*,[19]

$$Z(b, 1-b)=\omega^2\cdot\frac{\sqrt{x}}{1+x}\cdot F(x)\cdot F(1/x) \tag{3.56}$$

where $F(x)$ is an Eulerian product:

$$F(x)=\prod_{k=1,\dots,\infty}\left(1-\frac{x}{\omega^{8k-2}}\right)\cdot\left(1-\frac{x}{\omega^{8k+2}}\right)^{-1} \tag{3.57}$$

The expressions of the first X_n^H, Y_n, Z_n read, in terms of the variable x of (3.55),

$$X_1^H=\frac{x-\omega^2}{\omega(1+x)}, \qquad Y_1=\frac{\omega^4x-\omega^{-2}}{\omega(1+x)}, \qquad Z_1=\frac{\omega^3x+\omega^{-1}}{\omega(1+x)}$$

$$X_2^H = \frac{(x-\omega^2)(\omega^2 x - 1)}{(1+x)^2\,\omega^2}$$

$$Y_2 = \frac{(x-\omega^2)(\omega^6 x - \omega^{-4})}{(1+x)^2\,\omega^2}$$

$$Z_2 = \frac{(x-\omega^2)(\omega^5 x + \omega^{-3})}{(1+x)^2\,\omega^2}$$

$$X_3^H = \frac{(x-\omega^2)^2\,(x-\omega^{-2})(x-\omega^{-6})\,\omega^5}{(1+x)^3\,(\omega^6 x - 1)}$$

$$Y_3 = \frac{(x-\omega^2)^2\,(x-\omega^{-2})(x-\omega^{-14})\,\omega^6}{(1+x)^4}$$

$$Z_3 = \frac{(x-\omega^2)^2\,(x+\omega^{-12})(x-\omega^{-2})\,\omega^5}{(1+x)^4},... \tag{3.58}$$

It is clear that the polynomials X_n^H, Y_n, Z_n,... are closely related to the various factors occurring in the partition function (3.56), and more precisely in the "Eulerian" product (3.57).

When one considers weak-graph expansions[20] of this vertex model when it is no longer restricted to the Yang–Baxter integrability conditions (3.20), the "complexity" of the polynomials in b and c is very similar to that encountered with polynomials X_n^H, Y_n, Z_n,... seen as polynomials of the two variables b and c (see Appendix A). One can hope that these polynomials are well-suited to "decipher" the complexity encountered in weak-graph expansions of models which are not Yang–Baxter integrable.[21,22]

Therefore the following question arises: *Is it possible that the inversion trick*[4,19,23] *could, using such polynomials well-suited for the factorization analysis, yield an expansion in agreement with the weak-graph expansion?* This would open a new class of models in lattice statistical mechanics: models which are "computable" without being "Yang–Baxter integrable."[6] We will address this very important question in forthcoming publications. We, however, have a negative prejudice on this model, since the birational transformations K are not generically integrable [no foliation of the (b, c) plane in algebraic elliptic curves, chaotic behavior of the iteration of K

[6] Such models do exist: for instance, disorder solutions[24,25] provide some examples of "computable" models that are not Yang–Baxter-integrable. However, such disorder solutions correspond to dimensional reductions of the model. We are seeking here two-dimensional (or higher-dimensional) models with a genuine two-dimensional complexity.

outside the integrability conditions, exponential growth,...]. Therefore we will address this "computability-versus-Yang–Baxter-integrability" question on better suited models for which a foliation of the *whole parameter space* in terms of (algebraic) elliptic curves does exist.[7] The existence of these elliptic curves yields *analyticity properties* in *one variable* which *are known to be a key ingredient for the inversion trick to work.*[26–28]

3.6. Stroganov's Model for $q \geqslant 4$

These calculations can straightforwardly be generalized to arbitrary q, that is, $q^2 \times q^2$ matrices.

The way the exponential growth "degenerates" into a polynomial or linear growth [here, for model (3.28), the situation is even more drastic: there is no growth; seen as a homogeneous transformation, the degree of the N_n or D_n is 1] is exactly the same as for $q=3$. We have again factorizations (3.24) and (3.26), but now the generating functions $r(x)$ and $s(x)$ are, respectively, for arbitrary q,

$$r(x) = \frac{1+(q^2-2)\,x^2-x^3}{(1-x)[1-(q^2-2)\,x+x^2]} = 1 + \frac{x[(q^2-1)-x]}{(1-x)[1-(q^2-2)\,x+x^2]} \tag{3.59}$$

$$s(x) = \frac{1-x+(q^2-1)\,x^2-x^3}{(1-x)[1-(q^2-2)\,x+x^2]} = 1 + \frac{(q^2-2)\,x}{(1-x)[1-(q^2-2)\,x+x^2]} \tag{3.60}$$

The compatibility relation between factorizations (3.24) and (3.26) and the generic factorizations (3.13) reads

$$\beta_{n+2} = q^2(s_{n+1}+s_n) - (q^2-1)(r_n+r_{n-1}) + 1 \tag{3.61}$$

yielding for the associated generating functions $\alpha(x)$, $r(x)$, and $s(x)$

$$\beta(x) = (1+x)[q^2 x s(x) - (q^2-1)\,x^2 r(x)] + \frac{1}{1-x} - 1 - 2x + (q^2-2)\,x^2 + (q^2-1)\,x^3 \tag{3.62}$$

[7] We have called these models "quasiintegrable."[10] The most spectacular example of such a quasiintegrable, but not (generically) Yang–Baxter-integrable, model is the *16-vertex model.*[10]

Let us note that $s(x)$ has a remarkably simple form for $q^2=2$. Let us also note that the difference between $r(x)$ and $s(x)$ is quite simple:

$$r(x)=s(x)+\frac{x}{1-(q^2-2)\,x+x^2} \tag{3.63}$$

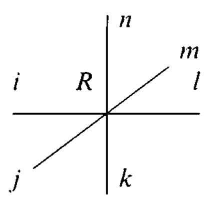
Of course this does not completely rule out integrability for q different from 2: many integrable subcases of the q^4-state vertex model are known in the literature, but the corresponding patterns of the R-matrices are very specific.[29]

In contrast, it will be seen in Section 5 that *polynomial growth* occurs when some of the "arrows" of the vertex models take two colors. It will be seen that this *polynomial growth* is closely related to the fact that the transformation K *can thus be represented as a shift on a Jacobian variety* naturally associated with K.

4. THREE-DIMENSIONAL VERTEX MODELS

4.1. Introduction

Let us now recall that, for a *three-dimensional cubic vertex model,*[8,9] the transposition t_1 associated with one of the three directions of the cubic lattice has already been introduced[8,9]:



The action of t_1 on the three-dimensional R-matrix is given by

$$(t_1 R)^{i_1 i_2 i_3}_{j_1 j_2 j_3} = R^{j_1 i_2 i_3}_{i_1 j_2 j_3} \tag{4.1}$$

with similar definitions for t_2 and t_3.[8,9]

Such a situation corresponds to $m=q^2$ in the framework described in Section 2. Of course, one can define t_2 and t_3 on this model because the q^2-dimensional space decomposes into the tensorial product of two q-dimensional spaces.

We will restrict consideration in this section to $q=2$; the results for an arbitrary value of q are given in Section 5. The analysis of the factorizations corresponding to the iterations of the transformation K for t_1 for a general

64-state three-dimensional model (generic 8×8 matrix) gives the following factorizations:

$$M_1 = K(M_0), \qquad f_1 = \det(M_0), \qquad f_2 = \frac{\det(M_1)}{f_1^4}$$
$$M_2 = \frac{K(M_1)}{f_1^3}, \qquad f_3 = \frac{\det(M_2)}{f_1^7 \cdot f_2^4}$$
$$M_3 = \frac{K(M_2)}{f_1^5 \cdot f_2^3}, \qquad f_4 = \frac{\det(M_3)}{f_1^8 \cdot f_2^7 \cdot f_3^4} \tag{4.2}$$
$$M_4 = \frac{K(M_3)}{f_1^6 \cdot f_2^5 \cdot f_3^3}, \qquad f_5 = \frac{\det(M_4)}{f_1^8 \cdot f_2^8 \cdot f_3^7 \cdot f_4^4}, \ldots$$

and, for arbitrary n, the following "*stringlike*" *factorizations*:

$$K(M_n) = M_{n+1} \cdot f_n^3 \cdot f_{n-1}^5 \cdot (f_{n-2} \cdot f_{n-3} \cdots f_1)^6 \tag{4.3}$$

$$\det(M_n) = f_{n+1} \cdot f_n^4 \cdot f_{n-1}^7 \cdot (f_{n-2} \cdot f_{n-3} \cdot f_{n-4} \cdots f_1)^8 \tag{4.4}$$

yielding

$$\hat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{(f_1 \cdot f_2 \cdots f_{n-1})^2 \cdot f_n \cdot f_{n+1}} \tag{4.5}$$

From the factorization (4.5), one easily gets a relation between the generating functions $\alpha(x)$ and $\beta(x)$:

$$(1+x)\,\alpha(x) - \frac{8(1+x^2)}{1-x}\,\beta(x) - 8 = 0 \tag{4.6}$$

leading to

$$\alpha(x) = \frac{8(1+x)^3}{(1-x)^4}, \qquad \beta(x) = \frac{8x}{(1-x)^3} \tag{4.7}$$

The "*right action*" of K also yields factorizations of f_1 and only f_1: one gets again Eqs. (3.3) (with of course different expressions for the μ_n and ν_n). These equations, combined with (4.6), give the following expressions for $\mu(x)$ and $\nu(x)$:

$$\mu(x) = \frac{x(1+x)(4-x)}{(1-x)^3}, \qquad \nu(x) = \frac{x(3+2x+x^2)}{(1-x)^4} \tag{4.8}$$

One notes that α_n and β_n are, respectively, *cubic and quadratic functions* of n [to be compared with (3.7)]:

$$\alpha_n = \tfrac{8}{3}(2n+1)(2n^2+2n+3), \qquad \beta_n = 4n(n+1) \tag{4.9}$$

At first sight it is amazing that such a polynomial growth occurs with involved "stringlike" factorizations, such as (4.3) *and* (4.4).

The occurrence of polynomial growth of the calculations of the iterations could correspond to situations where the algebraic varieties generated by K are Abelian varieties.[30] One knows that algebraic varieties having an *infinite set of automorphisms* cannot be of the so-called *general type.*[28], 8 This is the case here: we actually use the symmetries of the algebraic varieties (birational automorphisms) to visualize them.[5,6,8,9,11]

The analysis of the iterations of the transformation K has been performed in more detail for a particular 8×8 matrix corresponding to a three-dimensional generalization of the Baxter model.[8,9] This analysis shows that the orbits of the iterations lie, in this subcase,[8,9] on an *algebraic surface* given by the intersection of *quadrics.*[8,9] We will come back to this model in Section 4.1.1.

For the *general* 8×8-matrix considered here, the orbits do not lie on algebraic surfaces, but on higher-dimensional varieties.[8,9] Introducing *Plücker*-like variables closely related to the *minors* of the R-matrix,[1,3] here 4×4 minors, one can, for this three-dimensional vertex model, explicitly write down the equations of these algebraic varieties as the intersection of *quartics*. In fact, the analysis of these algebraic varieties[7–9] is difficult to perform: are these varieties *Abelian varieties*, or even *products of elliptic curves*,9 or any other algebraic varieties which are *not of the so-called "general type"*[28] (like $K3$ surfaces,10 ...)? We hope that the occurrence of polynomial growth of the associated iterations could help to clarify the kind of algebraic varieties associated with these birational transformations.

On the other hand, this could provide *a new way to analyze three- or higher-dimensional vertex models.* Of course, it is necessary to analyze *simultaneously* not only K, but also K_{t_2} and K_{t_3}, the birational transformations corresponding to the two other directions of the cubic lattice and to their associated partial transpositions t_2 and t_3.

8 Examples of algebraic varieties which are not of the general type are, for instance, in the case of surfaces, Abelian surfaces, hyperelliptic surfaces (surface fibered over $\mathbb{C}P_1$ by a pencil of elliptic curves), Enriques surfaces,... .

9 There exist some systematic procedures to see if an algebraic surface is a product of curves, but they are extremely difficult to implement.

10 The birational transformations considered here actually densify in a quite "uniform way" the algebraic surfaces we get (see Figs. 1a–1c): this seems to exclude automorphisms of $K3$ surfaces.

Let us try to better understand the relation between polynomial growth and the occurrence of various examples of algebraic varieties which not of the "general type."

For this purpose we now examine *different subcases* of this three-dimensional vertex model. A first subcase providing an example of a *quadratic growth associated with algebraic surfaces* which are the *product of two algebraic elliptic curves* is detailed in Appendix B1.

4.2. A Three-Dimensional Generalization of the Baxter Model

Another (less academic) example of "restricted factorization" corresponds to vertex models defined in refs. 8, 9, and 31, which can be seen as a three-dimensional generalization of the Baxter model. This model corresponds to the following K-compatible conditions:

$$R_{j_1 j_2 j_3}^{i_1 i_2 i_3} = R_{-j_1, -j_2, -j_3}^{-i_1, -i_2, -i_3} \tag{4.10}$$

$$R_{j_1 j_2 j_3}^{i_1 i_2 i_3} = 0 \qquad \text{if} \quad i_1 i_2 i_3 j_1 j_2 j_3 = -1 \tag{4.11}$$

Let us assume that the order for the "in" triplet (i_1, i_2, i_3) as well as the "out" triplet (j_1, j_2, j_3) is as follows:

$$\begin{aligned}
&[(1, 2, 3, 4, 5, 6, 7, 8)] \\
&\quad = [(+1, +1, +1), (+1, +1, -1), \\
&\qquad (+1, -1, +1), (+1, -1, -1), (-1, +1, +1), \\
&\qquad (-1, +1, -1), (-1, -1, +1), (-1, -1, -1)]
\end{aligned} \tag{4.12}$$

This order singles out direction 1, and is therefore well-suited to analyze the transformation K.

With this ordering, conditions (4.10) and (4.11) yield the following 8×8 matrix:

$$R^{3d} = \begin{pmatrix}
a & 0 & 0 & k & 0 & l & m & 0 \\
0 & b & c & 0 & n & 0 & 0 & d \\
0 & e & f & 0 & p & 0 & 0 & g \\
q & 0 & 0 & h & 0 & i & j & 0 \\
0 & j & i & 0 & h & 0 & 0 & q \\
g & 0 & 0 & p & 0 & f & e & 0 \\
d & 0 & 0 & n & 0 & c & b & 0 \\
0 & m & l & 0 & k & 0 & 0 & a
\end{pmatrix} \tag{4.13}$$

With another order for the rows and columns [namely $(1, 2, 3, 4, 5, 6, 7, 8) \to (1, 4, 6, 7, 8, 5, 3, 2)$], this 8×8 matrix can be seen as two identical 4×4 block matrices:

$$B^{3d} = \begin{pmatrix} a & k & l & m \\ q & h & i & j \\ g & p & f & e \\ d & n & c & b \end{pmatrix} \tag{4.14}$$

These two block matrices are respectively associated to the two "odd" and "even" subspaces:

$$[(+1, +1, +1), (+1, -1, -1), (-1, +1, -1), (-1, -1, +1)]$$

and

$$[(-1, -1, -1), (-1, +1, +1), (+1, -1, +1), (+1, +1, -1)]$$

With this new order the three directions 1, 2, and 3 are on the same footing: it is better suited to analyze the group generated by *all* the (four) inversion relations of this three-dimensional vertex model[8,9] (of course the transformation t_1 becomes a more involved permutation of the entries).

For this three-dimensional generalization of the eight-vertex model,[9,31] introducing the same f_n as the ones given by (4.2)–(4.4), one verifies the factorizations

$$\begin{aligned} &M_1 = K(M_0), \qquad f_1 = \det(M_0), \qquad f_2 = \frac{\det(M_1)}{f_1^4} \\ &M_2 = \frac{K(M_1)}{f_1^3}, \qquad f_3 = \frac{\det(M_2)}{f_1^7 \cdot f_2^4}, \qquad M_3 = \frac{K(M_2)}{f_1^6 \cdot f_2^3} \\ &f_4 = \frac{\det(M_3)}{f_2^7 \cdot f_3^4}, \qquad M_4 = \frac{K(M_3)}{f_2^6 \cdot f_3^3}, \qquad f_5 = \frac{\det(M_4)}{f_3^7 \cdot f_4^4}, \ldots \end{aligned} \tag{4.15}$$

Let us note that one has more factorizations than in the generic case (4.2). Moreover, for arbitrary n, one has the following factorizations, but now with a *fixed* number of polynomials f_n, instead of the "stringlike" factorizations (4.3) and (4.4):

$$K(M_n) = M_{n+1} \cdot f_n^3 \cdot f_{n-1}^6, \qquad \det(M_n) = f_{n+1} \cdot f_n^4 \cdot f_{n-1}^7 \tag{4.16}$$

yielding

$$\hat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_{n-1} \cdot f_n \cdot f_{n+1}} \tag{4.17}$$

Since the 8×8 matrix (4.13) is, after a relabeling of the rows and columns, *the direct product of two times the same* 4×4 *matrix*, and since the homogeneous transformation K acts in the same way on these two blocks, all these f_n *are exactly perfect squares.*

It is illuminating to see how factorizations like (4.2)–(4.5) become (4.15)–(4.17). One has the same first factorizations up to M_2 and f_3. They first become different with M_3, for which *one gets an extra factorization of* f_1. Obviously, in the factorization of f_4, one no longer has a factorization of f_1^8 [because an extra factorization of f_1 in all the entries of M_3 yields an extra factorization of f_1^8 in $\det(M_3)$]. These slight modifications, however, have the amazing consequence of changing the "stringlike" factorizations (4.3) and (4.5) *into factorizations with a fixed number of terms* [see relations (4.16) and (4.17)].

The generating functions $\alpha(x)$ and $\beta(x)$ satisfy

$$(1+x)\,\alpha(x) - 8(1+x+x^2)\,\beta(x) - 8 = 0 \tag{4.18}$$

leading to

$$\begin{aligned} \alpha(x) &= \frac{8(1+4x+7x^2)}{(1-x)^3}, \qquad & \beta(x) &= \frac{8x}{(1-x)^3} \\ \mu(x) &= \frac{x(1+x)(4-x)}{(1-x)^3}, \qquad & \nu(x) &= \frac{3x(1+2x)}{(1-x)^3} \end{aligned} \tag{4.19}$$

One notes that α_n and β_n are *both quadratic functions* of n:

$$\alpha_n = 8(6n^2+1), \qquad \beta_n = 4n(n+1) \tag{4.20}$$

One remarks that the two generating functions $\beta(x)$ and $\mu(x)$ *are the same as for the general* 8×8 matrix (see Section 4.1), the difference being in the α_n or the ν_n [or equivalently in the generating functions $\alpha(x)$ and $\nu(x)$]: the *cubic* growth of the α_n or the ν_n [see relation (4.9)] is replaced by a *quadratic growth* [see relation (4.20)].

In fact this modification of a "stringlike" factorization into factorizations with a *fixed* number of terms is not as drastic as it looks at first sight. Let us, for instance, define, for an 8×8 matrix of the form (4.13), the variables f_n^{string} and the successive matrices M_n^{string} from the "stringlike" factorization relations (4.3) and (4.5), which are valid for general 8×8

matrices, and therefore, *a fortiori*, for matrices of the form (4.13). We have just seen that an extra factorization occurs for model (4.13) (M_3,...). It is amusing to remark *that the variables* f_n^{string} *defined from* (4.3) *and* (4.5) *and the variables* f_n *defined from* (4.16) *actually coincide*! This can be proved recursively. Let us denote by y_n the multiplicative factor between M_n [defined by (4.16)] and M_n^{string}: $M_n^{\mathrm{string}} = y_n \cdot M_n$. One immediately gets, from (4.3) and (4.16), the following relations:

$$\frac{1}{y_{n+1}} = \frac{(f_1 \cdot f_2 \cdots f_{n-2})^6}{f_{n-1} \cdot y_n^2} \quad \text{and} \quad \frac{f_n}{f_n^{\mathrm{string}}} = \frac{f_1 \cdot f_2 \cdots f_{n-3}}{y_{n-1}} \tag{4.21}$$

It is then simple to show recursively that $y_n = f_1 \cdot f_2 \cdots f_{n-2}$ and therefore that $f_n = f_n^{\mathrm{string}}$. This means that this stringlike factorization may be seen, to some extent, just as a "propagation" of the extra factorization occurring with M_3. This explains that the generating functions $\mu(x)$, $\beta(x)$ *are actually identical* for matrices of the form (4.13) and for general 8×8 matrices [see (4.7), (4.8), and (4.19)].

If one relaxes the spin reversal constraint (4.10) {for instance, just relaxing the equality between $M_0[1,1]$ and $M_0[8,8]$ in (4.13)}, which means that, after relabeling, the 8×8 matrix can be written as two *nonidentical* 4×4 block matrices, *one gets back to the above-detailed "stringlike" factorizations* of Section 4.1. Of course, if one relaxes the "charge-conservation" constraint (4.11) [for instance, just make $M_0[1,2]$ nonzero in (4.13)], one also gets back to the above-detailed "stringlike" factorizations of Section 4.1.

Let us note that the orbits of K can be shown to yield algebraic varieties in $\mathbb{C}P_{15}$ given by the *intersection of quadrics*.[(1), 11]

4.2.1. A Nine-Parameter Three-Dimensional Generalization of the Baxter Model. The visualization of the orbits of K has been performed in refs. 8 and 9 for the particular subcase which amounts to imposing, together with (4.10) and (4.11), that the matrix R of (4.13) be *symmetric*:

$$R^{i_1 i_2 i_3}_{j_1 j_2 j_3} = R^{j_1, j_2, j_3}_{i_1, i_2, i_3} \tag{4.22}$$

In this subcase one clearly gets *surfaces* and it has been shown that these surfaces are *algebraic surfaces* given by the *intersecton of quadrics*.[(1,9,31), 12] The explicit expressions of these quadrics have been written down in the

[11] The occurrence of *quadrics* is closely related[(1,3)] to the occurrence of 4×4 matrices like (4.14) for model (4.13).

[12] When conditions (4.10) and (4.11) are relaxed one no longer gets (algebraic) surfaces, but higher-dimensional varieties.

particular subcase of the model defined by (4.10) and (4.11) together with (4.22).[1,9,31] Condition (4.22) is clearly preserved by the transformation I and $t_1 \cdot t_2 \cdot t_3$ (the matrix inversion and the matrix transposition). More rearkably, condition (4.22) is actually *preserved by the three other inversions*[8,9] of this three-dimensional vertex model, namely $I_1 = t_1 \cdot I \cdot t_2 \cdot t_3$, $I_2 = t_2 \cdot I \cdot t_1 \cdot t_3$, and $I_3 = t_3 \cdot I \cdot t_1 \cdot t_2$. This is a consequence of the fact that condition (4.22) is preserved by the partial transpositions t_1, t_2, and t_3. With this last condition the three-dimensional vertex model looks even more closely like a generalization in three dimensions of the symmetric eight-vertex Baxter model.[9,8] When condition (4.22) is satisfied together with conditions (4.10) and (4.11), the two identical block matrices (4.14) depend only on *ten* homogeneous parameters. Using the notations introduced in ref. 9 or ref. 31, one can introduce ten homogeneous parameters:

$$B^{3d} = \begin{pmatrix} a & k & l & m \\ q & h & i & j \\ g & p & f & e \\ d & n & c & b \end{pmatrix} = \begin{pmatrix} a & d_1 & d_2 & d_3 \\ d_1 & b_1 & c_3 & c_2 \\ d_2 & c_3 & b_2 & c_1 \\ d_3 & c_2 & c_1 & b_3 \end{pmatrix} \tag{4.23}$$

Of course, the transformation t_1, which is the block transposition of the two off-diagonal 4×4 matrices, and also transformations t_2 and t_3 become, as a consequence of the relabeling, new permutations of the entries of this 4×4 matrix B^{3d},[31]

$$t_i\!: \quad c_j \leftrightarrow d_j, \quad c_k \leftrightarrow d_k, \qquad (i, j, k) = (1, 2, 3) \tag{4.24}$$

Actually, and quite remarkably, there exist four quantities which are invariant by *all four generating involutions* I, I_1, I_2, I_3 and therefore the whole group Γ_{3D} they generate. Let us recall the results of ref. 31.

Let us introduce

$$ab_1 + b_2 b_3 - c_1^2 - d_1^2, \qquad c_2 d_2 - c_3 d_3 \tag{4.25}$$

and the polynomials obtained by permutations of 1, 2, and 3. They form a five-dimensional space of polynomials. Any ratio of the five independent polynomials is invariant under all four generating involutions I, I_1, I_2, I_3. The parameter space $\mathbb{C}P_9$ is thus foliated by *five-dimensional* algebraic varieties invariant under *the whole group* Γ_{3D}:

$$\frac{P_i(a, \ldots, d_3)}{Q_i(a, \ldots, d_3)} = \text{const} \tag{4.26}$$

where P_i and Q_i are chosen among the quadratic polynomials (4.25) and the one deduced by permutations of directions 1, 2, and 3.

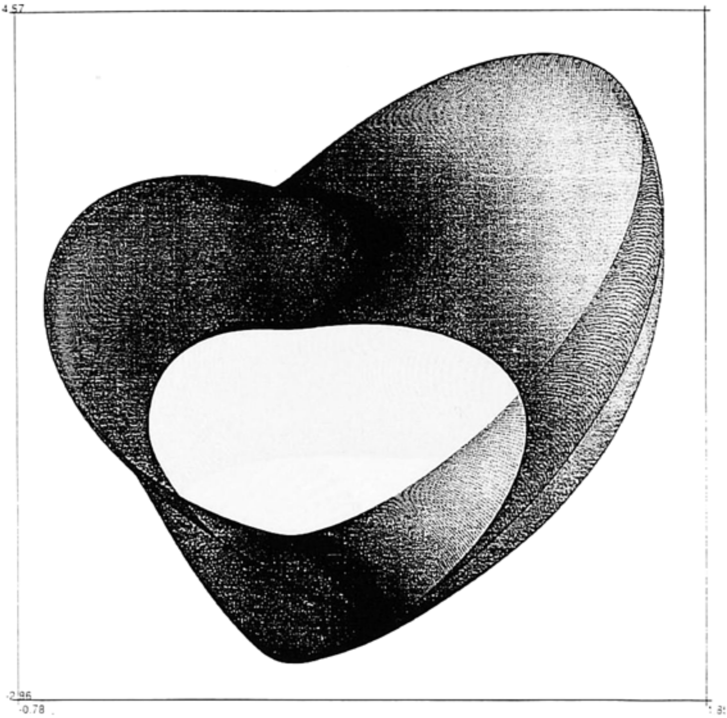


Fig. 1. Two-dimensional projection of the iteration of the transformation K^2 acting in the nine-dimensional parameter space of the three-dimensional vertex model (4.13) corresponding to a symmetric 4×4 matrix (4.23).

Considering the subgroup of Γ_{3D} generated by two involutions among these four, or equivalently considering the iteration of K only,[13] one can show that the orbits of this transformation are *algebraic surfaces* given by intersection of quadrics.[1,9,11,13]

These additional quadrics have been written explicitly[31],[14]

$$ab_1 - b_2 b_3 - c_1^2 - d_1^2, \qquad (a + b_1)\, c_1 - d_2 d_3 - c_2 c_3$$
$$(b_2 + b_3)\, d_1 - d_2 c_3 - d_3 c_2 \tag{4.27}$$

[13] Since I commutes with the matrix transposition $t_1 \cdot t_2 \cdot t_3$, the product of two inversions, for instance, $I_1 \cdot I = t_1 \cdot I \cdot t_1 \cdot t_2 \cdot t_3 \cdot I = K^2 \cdot t_1 \cdot t_2 \cdot t_3$ is equivalent, up to the matrix transposition $t_1 \cdot t_2 \cdot t_3$, to K^2.

[14] One should note a misprint in ref. 31: one should read $ab_1 - b_2 b_3 - c_1^2 - d_1^2$ instead of $ab_1 - b_2 b_3 - c_1^2 + d_1^2$.

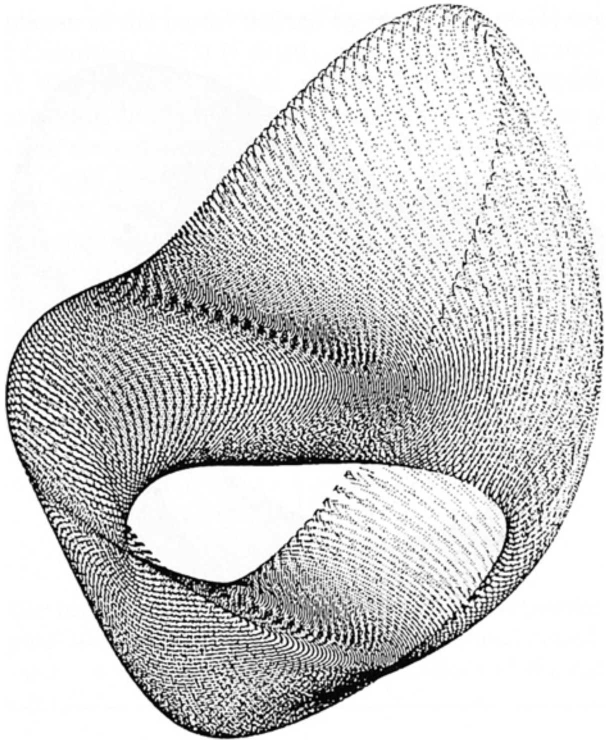


Fig. 1 (*continued*)

Figures 1a–1c show clearly that the orbits of K are (algebraic) *surfaces*. These orbits strongly suggest an interpretation in terms of curves winding around a two-dimensional torus in the generic "incommensurate" situation.

In contrast with the situation encountered in Section 3.1 [see relation (3.8)], the successive iterates of M_0 live in the *whole nine-dimensional* affine matrix space (4.23):

$$K^{2n}(M_0) = a_0^{(n)} \cdot M_0 + a_1^{(n)} \cdot M_2 \cdots + a_8^{(n)} \cdot M_{16} + a_9^{(n)} \cdot M_{18} \qquad (4.28)$$

The visualization of the orbits of K has also been performed when *one relaxes the matrix symmetry condition* (4.22). *One no longer finds surfaces.* Figures 2a–2c illustrate such a situation. Figure 2a corresponds to an orbit for an initial matrix "almost" symmetric (symmetric up to 10^{-6}). This first figure, which corresponds to a very small asymmetry of the initial matrix,

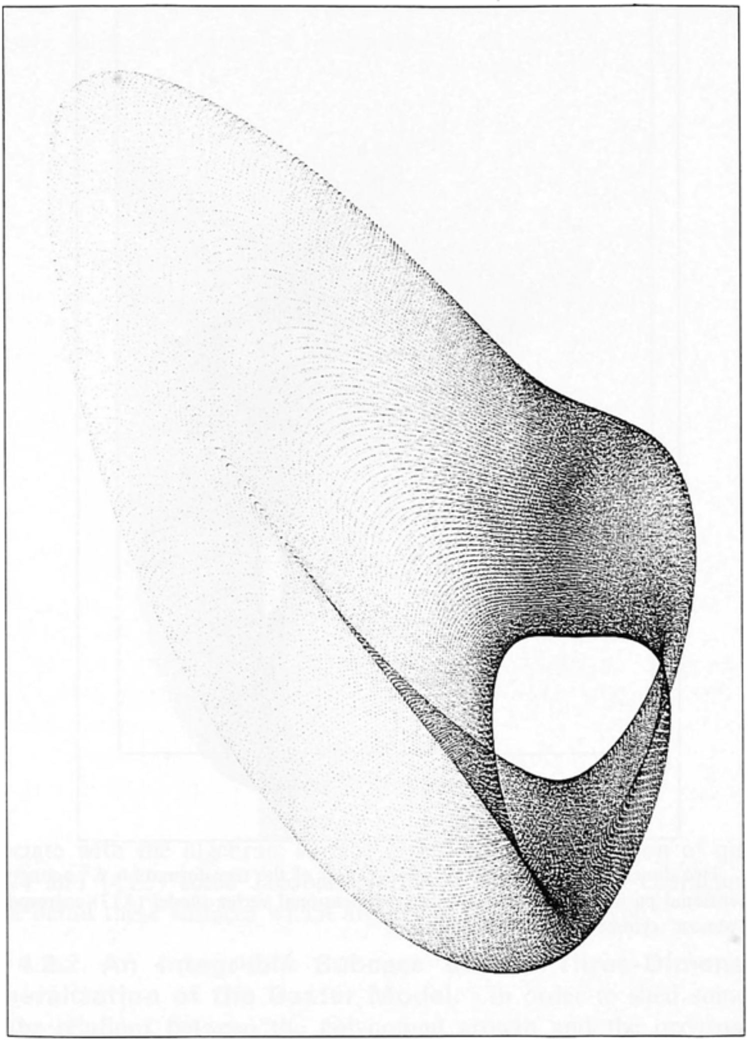


Fig. 1 (*continued*)

may look similar (at least for the first 10^5 iterations) to Figs. 1a–1c. In fact, one can see in Fig. 2a that the density of points is more "fuzzy" compared to Figs. 1a–1c, which suggests a curve moving on a two-dimensional torus. The density of points of Figs. 2b and 2c clearly corresponds to the projection of points living in algebraic varieties of dimension greater than two.

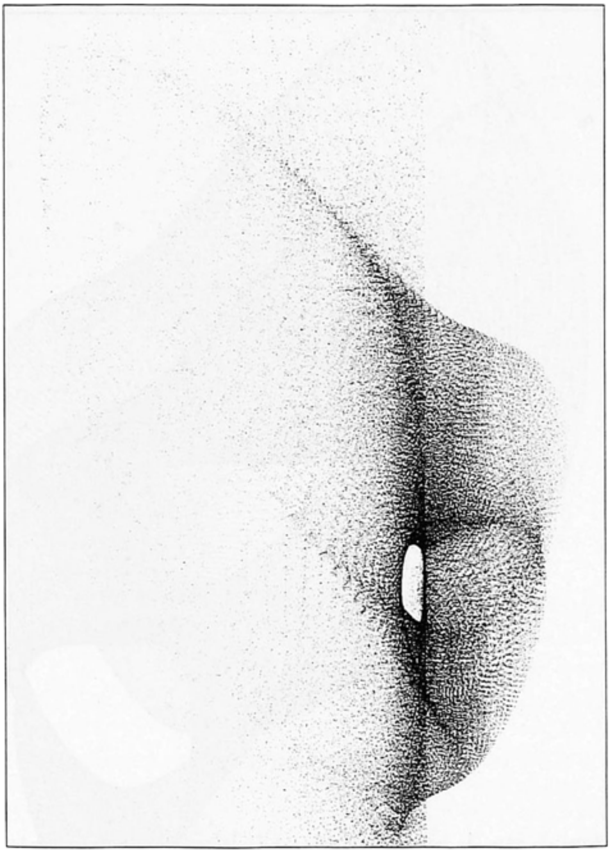


Fig. 2. Two-dimensional projection of the iteration of the transformation K^2 acting in the 15-dimensional parameter space of the three-dimensional vertex model (4.13) corresponding to an "almost" symmetric 4×4 matrix (4.23).

These results have to be compared with the one given by Korepanov[32] or the one described in Appendix C. The fact that a polynomial growth occurs when some of the "arrows" in the vertex models take two colors and that exponential growth (generically) occurs when the number of colors of the "arrows" is no longer 2 (see Section 3.2) deserves some comment: this *polynomial growth* is related to the fact that the transformation K *can be represented as a shift on a Jacobian variety* naturally associated with K. We previously recalled that algebraic varieties having an *infinite set of automorphisms* cannot be of the so-called *general type*.[28] The fact that one can

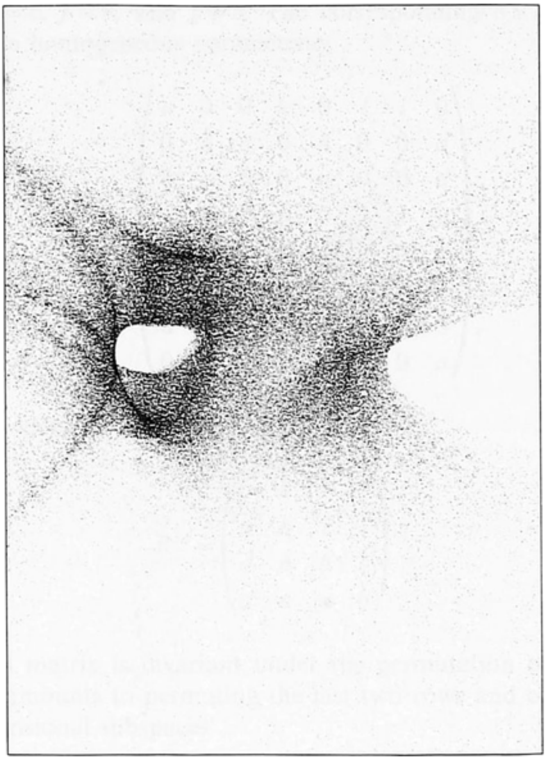


Fig. 2 (*continued*)

associate with the algebraic surface given by the intersection of quadrics (4.27) and (4.25) some Jacobian variety should help to characterize in more detail these surfaces which are not of the general type.[15]

4.2.2. An Integrable Subcase of the Three-Dimensional Generalization of the Baxter Model. In order to shed some light on the relations between the polynomial growth and the occurrence of *algebraic varieties which are not of the so-called "general" type*[28] (Abelian varieties, products of elliptic curves,...), let us consider a situation for which *elliptic curves occur.* At this point, it is worth recalling that, for particular patterns of the three-dimensional generalization of the Baxter model considered in Section 4.2.1 [conditions (4.11), (4.10) together with *the additional condition* (4.22)], the iteration of K (or $\hat{K}$) can actually *yield*

[15] Note that the space where this Jacobian variety lives is, in general, not the same as the parameter space $\mathbb{C}P_{q^2-1}$ where these algebraic varieties generated by K live.

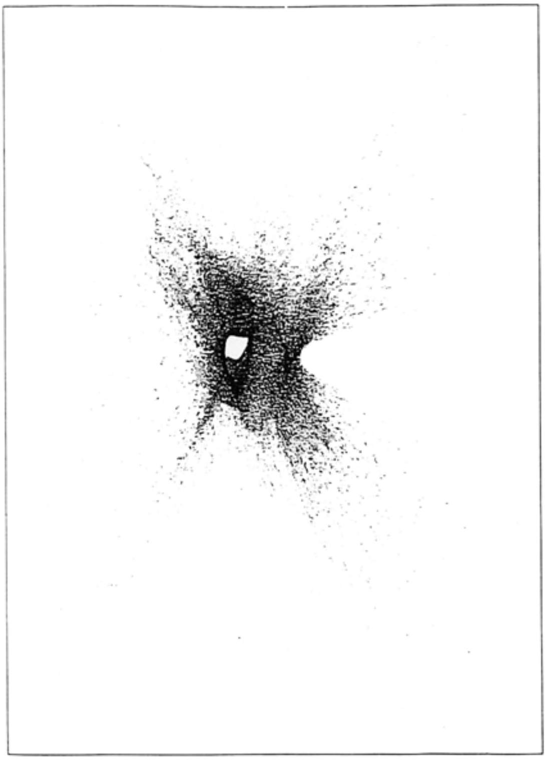


Fig. 2 (*continued*)

elliptic curves.[8,9] These particular patterns amount to imposing that the initial matrix (and therefore the successive matrices M_n) is *invariant under the permutation of the two directions* 1 and 2. In fact we will see in this section that there is no need to impose the matrix symmetry condition (4.22) to get integrable subcases of (4.13).

With the previous order (4.12) for the rows and columns of the 8×8 matrix, these additional conditions read

$$R[1,6]=R[1,7], \qquad R[2,8]=R[3,8], \qquad R[2,5]=R[3,5]$$

$$R[4,6]=R[4,7], \qquad R[2,2]=R[3,3], \qquad R[2,3]=R[3,2]$$

Recalling matrix (4.13) and its notations, this symmetry between directions 2 and 3 yields the following additional equalities among the entries: $m=l$,

$j=i$, $g=d$, $e=c$, $f=b$, and $p=n$. The corresponding 8×8 matrix thus depends on *ten* homogeneous parameters:

$$R^{3d}=\begin{pmatrix} a & 0 & 0 & k & 0 & l & l & 0 \\ 0 & b & c & 0 & n & 0 & 0 & d \\ 0 & c & b & 0 & n & 0 & 0 & d \\ q & 0 & 0 & h & 0 & i & i & 0 \\ 0 & i & i & 0 & h & 0 & 0 & q \\ d & 0 & 0 & n & 0 & b & c & 0 \\ d & 0 & 0 & n & 0 & c & b & 0 \\ 0 & l & l & 0 & k & 0 & 0 & a \end{pmatrix} \tag{4.29}$$

or on the two identical 4×4 block matrices (4.14):

$$B^{3d}=\begin{pmatrix} a & k & l & l \\ q & h & i & i \\ d & n & b & c \\ d & n & c & b \end{pmatrix} \tag{4.30}$$

This 4×4 matrix is invariant under the permutation of directions 2 and 3, which amounts to permuting the last two rows and columns of the two four-dimensional subspaces

$$[(+1,\ +1,\ +1), (+1,\ -1,\ -1), (-1,\ +1,\ -1), (-1,\ -1,\ +1)]$$

and

$$[(-1,\ -1,\ -1), (-1,\ +1,\ +1), (+1,\ -1,\ +1), (+1,\ +1,\ -1)]$$

Imposing the additional constraints (4.29), one remarks that the factorizations (4.16) and (4.17) are slightly, but definitely, modified as follows:

$$\begin{gathered} M_1=K(M_0), \qquad f_1=\det(M_0), \qquad f_2=\frac{\det(M_1)}{f_1^5} \\ M_2=\frac{K(M_1)}{f_1^4}, \qquad f_3=\frac{\det(M_2)}{f_1^3\cdot f_2^5}, \qquad M_3=\frac{K(M_2)}{f_1^2\cdot f_2^4} \\ f_4=\frac{\det(M_3)}{f_1^7\cdot f_2^3\cdot f_3^5}, \qquad M_4=\frac{K(M_3)}{f_1^6\cdot f_2^2\cdot f_3^4}, \qquad f_5=\frac{\det(M_4)}{f_2^7\cdot f_3^3\cdot f_4^5},\ldots \end{gathered} \tag{4.31}$$

and for arbitrary n

$$K(M_n) = M_{n+1} \cdot f_n^4 \cdot f_{n-1}^2 \cdot f_{n-2}^6, \qquad \det(M_n) = f_{n+1} \cdot f_n^5 \cdot f_{n-1}^3 \cdot f_{n-2}^7 \tag{4.32}$$

yielding

$$\hat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_{n-2} \cdot f_{n-1} \cdot f_n \cdot f_{n+1}} \tag{4.33}$$

Note that the "universal" relation (4.5) is actually modified for subcase (4.29). The new polynomials, defined in this restricted (integrable) subcase (4.29), can actually be shown to satisfy nonlinear recursion relations. Since the f_n are *perfect squares*, one can introduce their square roots $f'_n = f_n^{1/2}$. Remarkably, these polynomials f'_n *satisfy the same hierarchy of recursion relations as for the 16-vertex model* (see Section 3.1 and ref. 1):

$$\frac{f'_n(f'_{n+3})^2 - f'_{n+4}(f'_{n+1})^2}{f'_{n-1}f'_{n+3}f'_{n+4} - f'_nf'_{n+1}f'_{n+5}} = \frac{f'_{n-1}(f'_{n+2})^2 - f'_{n+3}(f'_n)^2}{f'_{n-2}f'_{n+2}f'_{n+3} - f'_{n-1}f'_nf'_{n+4}} \tag{4.34}$$

or

$$\frac{f'_{n+1}(f'_{n+4})^2 f'_{n+5} - f'_{n+2}(f'_{n+3})^2 f'_{n+6}}{(f'_{n+2})^2 f'_{n+3}f'_{n+7} - f'_nf'_{n+4}(f'_{n+5})^2}$$
$$= \frac{f'_{n+2}(f'_{n+5})^2 f'_{n+6} - f'_{n+3}(f'_{n+4})^2 f'_{n+7}}{(f'_{n+3})^2 f'_{n+4}f'_{n+8} - f'_{n+1}f'_{n+5}(f'_{n+6})^2} \tag{4.35}$$

These recursion relations are known to yield *elliptic curves.*[1-3] The generating functions $\alpha(x)$, $\beta(x)$, $\mu(x)$, and $\nu(x)$ read

$$\alpha(x) = \frac{8(1+5x+3x^2+7x^3)}{(1+x)(1-x)^3}, \qquad \beta(x) = \frac{8x}{(1+x)(1-x)^3} \tag{4.36}$$

$$\mu(x) = \frac{x(5+2x^2-x^3)}{(1+x)(1-x)^3}, \qquad \nu(x) = \frac{2x(2+x+3x^2)}{(1+x)(1-x)^3} \tag{4.37}$$

The integrability of this subcase (4.29) is thus associated with the occurrence of one more singularity [compare with (4.7) or (4.19)].

In contrast with the situation we had in Section 3.1 [see relation (3.8)], the successive iterates of M_0 belong, for this subcase (4.29), to a *seven*-dimensional affine subspace of the *nine*-dimensional affine matrix space [(4.29) depends on *ten homogeneous parameters*]

$$K^{2n}(M_0) = a_0^{(n)} \cdot M_0 + a_1^{(n)} \cdot M_2 \cdots + a_7^{(n)} \cdot M_{14} \tag{4.38}$$

which is a codimension-two subspace of the space where the matrices M_0 live.

The equations of these elliptic curves can be simply written down as the intersection of the quadrics and of the hyperplanes preserved by K^2 [see (4.29)]. For the model (4.23), analyzed in refs. 8 and 9 (see Section 4.2.1), which amounts to imposing the Boltzmann matrix to be symmetric [condition (4.22)], these quadrics are (4.25) and (4.27), and these hyperplanes read, with notations (4.23),

$$b_2 = b_3, \qquad c_2 = c_3, \qquad d_2 = d_3 \tag{4.39}$$

4.2.3. A Three-Dimensional Generalization of the Six-Vertex Model. Another example of "restricted factorization" corresponds to the vertex model defined in refs. 8, 9, and 31, which can be seen as a three-dimensional generalization of the *six-vertex model.*[31] The recursion relations are a little bit more involved compared to the previous examples, but still yield polynomial growth with similar generating functions. Detailed calculations are given in Appendix B2.

5. GENERALIZATION TO d-DIMENSIONAL VERTEX MODELS AND MONODROMY MATRICES

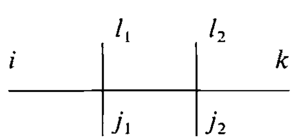
We consider the matrix (2.3) for an arbitrary m-dimensional space when there are only two spin states in direction 1, that is, $q = 2$.

The transposition t_1 amounts to permuting two off-diagonal $m \times m$ submatrices of this $2m \times 2m$ R-matrix,[16]

$$t_1\colon \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{5.1}$$

where A, B, C, and D are $m \times m$ matrices.

Such a formalism can represent many different situations encountered in lattice statistical mechanics for vertex models. Namely, it can describe d-dimensional vertex models as well as *monodromy matrices.*[33] These monodromy matrices can be written as (5.1), where the matrices A, B, C, and D are now $2^N \times 2^N$ matrices. Let us give a pictoral representation of the two-site case ($N = 2$):



(5.2)

[16] This problem exactly corresponds to the one considered by Korepanov.[32]

For d-dimensional vertex models, m is equal to 2^{d-1}. In this case one can also consider transpositions $t_2, t_3,..., t_{d-1}$,[8,9] like the t_1 associated with the $d-1$ other directions, and of course one obtains similar results for all the t_i.

For arbitrary m (equal to 2^{d-1} or not), the analysis of the factorizations of the iterations of the transformation K yields

$$M_1 = K(M_0), \qquad f_1 = \det(M_0), \qquad f_2 = \frac{\det(M_1)}{f_1^{2(m-2)}}, \qquad M_2 = \frac{K(M_1)}{f_1^{2m-5}}$$

$$f_3 = \frac{\det(M_2)}{f_1^7 \cdot f_2^{2(m-2)}}, \qquad M_3 = \frac{K(M_2)}{f_1^5 \cdot f_2^{2m-5}}, \qquad f_4 = \frac{\det(M_3)}{f_1^{4(m-2)} \cdot f_2^7 \cdot f_3^{2(m-2)}} \tag{5.3}$$

$$M_4 = \frac{K(M_3)}{f_1^{2(2m-5)} \cdot f_2^5 \cdot f_3^{2m-5}}, \qquad f_5 = \frac{\det(M_4)}{f_1^8 \cdot f_2^{4(m-2)} \cdot f_3^7 \cdot f_4^{2(m-2)}},...$$

and, for arbitrary n, the following "stringlike" factorizations:

$$K(M_n) = M_{n+1} \cdot f_n^{2m-5} \cdot f_{n-1}^5 \cdot f_{n-2}^{2(2m-5)} \cdot f_{n-3}^6 \cdot f_{n-4}^{2(2m-5)} \cdot f_{n-5}^6 \cdots \tag{5.4}$$

$$\det(M_n) = f_{n+1} \cdot f_n^{2(m-2)} \cdot f_{n-1}^7 \cdot f_{n-2}^{4(m-2)} \cdot f_{n-3}^8 \cdot f_{n-4}^{4(m-2)} \cdot f_{n-5}^8 \cdot f_{n-6}^{4(m-2)} \cdots \tag{5.5}$$

Equations (5.4) and (5.5) yield the following relation *independent of m*:

$$\hat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{(f_1 \cdot f_2 \cdots f_{n-1})^2 \cdot f_n \cdot f_{n+1}} \tag{5.6}$$

Equation (5.6) gives again a generalization of Eq. (4.6) for arbitrary m:

$$(1+x) \cdot \alpha(x) - \frac{2m(1+x^2)}{1-x} \cdot \beta(x) - 2m = 0 \tag{5.7}$$

From (5.4) and (5.5) and also from (3.3), which is indeed valid, one gets

$$\alpha(x) = 2m \frac{(1-x)^4 + 2mx(1+x^2)}{(1+x)(1-x)^4}, \qquad \beta(x) = \frac{2mx}{(1-x)^3} \tag{5.8}$$

$$\mu(x) = \frac{x(2m-4+3x-x^2)}{(1-x)^3}, \qquad \nu(x) = \frac{x[(2m-5)(1+x^2)+5x+x^3]}{(1-x)^4\,(1+x)} \tag{5.9}$$

Let us underline that, for $m=4$, one recovers (4.7) and (4.8) taking the $m=4$ limit of expressions (5.8). One also recovers factorizations (4.3) and (4.4) taking the $m=4$ limit of factorizations (5.4) and (5.5).

5.1. Comments: Pre-Bethe Ansatz and Gauge Transformations

All the examples of vertex models given here show that the number of colors two, for the arrows of the vertex models, *plays a special role for the occurrence of polynomial growth.*

This property is related to the fact that the transformation K can be represented as a shift on a Jacobian variety C^g/Γ. Where does this Jacobian variety come from?

Actually, for all the vertex models for which the transposition t_1 can be represented as (5.1) (namely monodromy matrices as in Section 5 or d-dimensional vertex models with arrows taking two colors,...) one can associate (see Appendix C) an algebraic curve of equation

$$\det(Ap' - C - Dp + pp'B) = 0 \tag{5.10}$$

As a byproduct, this provides a *canonical Jacobian variety for such vertex models*, namely the Jacobian variety associated with curve (5.10). This procedure, which associates with an R-matrix the algebraic curve (5.10), originates from a key "factorization" relation closely related to the action of the birational transformations K, namely the "pre-Bethe Ansatz" condition.[10,34,35] More details are given in Appendix C.

In fact, it will be shown in forthcoming publications that one can prove the polynomial growth of the calculations[36] when the transformation K can be represented as a *shift* on a Jacobian variety C^n/Γ. This enables us to better understand *why the number of colors two* for the arrows of the vertex models *plays such a special role for the occurrence of polynomial growth.*

Let us also note that the curve (5.10) has appropriate invariance properties with respect to "gaugelike" transformations generalizing the weak-graph transformations.[20] Taking into account the relation between the transformation K and the "pre-Bethe Ansatz" condition[17] and therefore the algebraic curve (5.10) or its associated Jacobian variety, it is not surprising to see that these "gaugelike" transformations are also symmetries of the transformation K, *compatible with the factorizations* (5.4) and (5.5) (see Appendix D).

[17] The relevance of the "pre-Bethe Ansatz" condition is not clear as far as, for instance, calculating the partition function is concerned, but its significance for the discrete symmetries considered here is well established[10]: this is a consequence of the compatibility of this condition with the transformation I together with the partial transposition t_1 or t_2.

6. SOME COMMENTS ON THE GENERATING FUNCTIONS: FROM VERTEX TO SPIN MODELS

For all the birational transformations described here, one remarks that one always has the three following factorization relations:

$$\det(M_n) = f_{n+1} \cdot f_n^{\zeta_1} \cdot f_{n-1}^{\zeta_2} \cdot f_{n-2}^{\zeta_3} \cdot f_{n-3}^{\zeta_4} \cdot f_{n-4}^{\zeta_5} \cdots f_1^{\zeta_n} \tag{6.1}$$

$$K(M_n) = M_{n+1} \cdot f_n^{\eta_0} \cdot f_{n-1}^{\eta_1} \cdot f_{n-2}^{\eta_2} \cdot f_{n-3}^{\eta_3} \cdot f_{n-4}^{\eta_4} \cdots f_1^{\eta_{n-1}} \tag{6.2}$$

$$\hat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{f_{n+1} \cdot f_n^{\rho_1} \cdot f_{n-1}^{\rho_2} \cdot f_{n-2}^{\rho_3} \cdot f_{n-3}^{\rho_4} \cdots f_1^{\rho_n}} \tag{6.3}$$

Let us introduce a new generating function for the ζ_n:

$$\zeta(x) = 1 + \zeta_1 x + \zeta_2 x^2 + \zeta_3 x^3 + \cdots \tag{6.4}$$

With this new generating function $\zeta(x)$, relation (6.1) simply reads

$$x\alpha(x) = \zeta(x) \cdot \beta(x) \tag{6.5}$$

One can also introduce generating functions for the η_n and ρ_n:

$$\eta(x) = \eta_0 + \eta_1 x + \eta_2 x^2 + \eta_3 x^3 + \cdots \tag{6.6}$$

$$\rho(x) = 1 + \rho_1 x + \rho_2 x^2 + \rho_3 x^3 + \cdots \tag{6.7}$$

One gets from relation (6.3) the following relation between $\alpha(x)$, $\beta(x)$, and $\rho(x)$:

$$N + N\rho(x) \cdot \beta(x) = (1 + x) \cdot \alpha(x) \tag{6.8}$$

which generalizes Eqs. (5.7) for an arbitrary $N \times N$ matrix.

Many more relations can be obtained among these various generating functions $\alpha(x)$, $\beta(x)$, $\mu(x)$, $\nu(x)$,... .[1,2]

Among these more or less involved generating functions, it appears that two generating functions are especially simple, namely $\beta(x)$ and particularly $\rho(x)$. Let us give here the explicit expressions of $\rho(x)$ for various vertex models considered in this paper. For t_1 for 4×4, as well as $q^2 \times q^2$, matrices (see Sections 3.1 and 3.2) the expression for $\rho(x)$ is

$$\rho(x) = 1 + x^2 \tag{6.9}$$

while for t_1 for $2^d \times 2^d$ (or $2m \times 2m$) matrices [see Sections 4 and 5 and Eq. (5.7)] $\rho(x)$ is given by

$$\rho(x) = \frac{1 + x^2}{1 - x} \tag{6.10}$$

These two kinds of generalizations of the transformation t_1 for arbitrary size of the matrices are of a quite different nature. In particular, the size dependence of the generating functions [in particular, $\beta(x)$] is quite different. It is simpler for the generalizations described in Sections 4 and 5 [compare, for instance, the expression for $\beta(x)$ in (5.8) and (3.17)]. Note, however, that $\rho(x)$ is remarkably simple for both kinds of size generalizations, since it has zeros or poles *only on the unit circle and it is actually independent of the matrix size.*

The polynomial, or exponential, growth of the calculations of the iterations is made clear for the singularities of the other generating functions $\alpha(x)$, $\beta(x)$,..., or even the generating functions $\eta(x)$ and $\zeta(x)$. This provides a condition for the *polynomial growth* of the calculations, which can therefore be checked quickly from relations (6.1), (6.3).

The *polynomial growth* of the calculations corresponds to poles on the unit circle. In fact in all the examples we have introduced[1-3] one only gets Nth *roots of unity* in the denominators of the rational functions $\alpha(x)$, $\beta(x)$,..., and, most of the time, only $x = \pm 1$ singularities. We have obtained very few Nth roots of unity different from $x = \pm 1$. One example corresponds to an integrable subcase of a birational transformation (denoted class IV in ref. 3), which yields, when restricted to this integrable subcase,[2]

$$\beta(x) = \frac{4x}{(1-x)(1-x^2)(1-x^3)} \tag{6.11}$$

Another interesting example corresponds to a (*six-state chiral*) edge spin model[5,6] for which a foliation in terms of elliptic functions exists.[5] The analysis developed here, or in refs. 1–3 for vertex models, has to be slightly modified[12] when considering edge spin models or IRF models. However, it is worth noticing that one gets, for these integrable birational mappings, a generating function for the growth of the calculations where *third and fourth roots of unity occur*[14]:

$$G(x) = \frac{(1+x+2x^2+x^3+2x^4)(1+2x+2x^2+2x^3)}{(1-x)(1-x^3)(1-x^4)} \tag{6.12}$$

The growth of these coefficients, that is, the growth of the degree of the successive iterations, is dominated by the coefficients of the expansion of

$$G_{\mathrm{dom}}(x) = \frac{49}{12(1-x)^3} \tag{6.13}$$

which grows like $49(n+1)(n+2)/24$. Another example is a *five-state* Potts model[6,37] (symmetric and cyclic 5×5 matrices), which yields integrable birational mappings (a foliation of $\mathbb{C}P_2$ in algebraic elliptic curves).[6] For this edge spin model the generating function for the growth of the calculations is given by[14],[18]

$$G_{\text{Potts}}(x) = \frac{(1+x+2x^2)^2}{(1-x)^2\,(1-x^3)} \tag{6.14}$$

The similarity with expression (6.12) is striking. The growth of these coefficients is dominated by the coefficients of the expansion of

$$G_{\text{dom}}^{\text{Potts}}(x) = \frac{16}{3(1-x)^3} \tag{6.15}$$

Another interesting example of a spin model is the q-state standard scalar Potts model on a triangular lattice with two- and three-site interactions, introduced by Baxter *et al.*[17] (BTA). Because of the three-site interactions on the up-pointing triangles, this model is not an edge spin model. It can also be represented as a vertex model on a triangular lattice.[17] It has been shown that the symmetry group generated by the inversion relations yields birational representations of hyperbolic Coxeter groups.[38] Some of the generators of this group have been shown to yield algebraic elliptic curves and even rational curves.[38] Let us consider the factorizations corresponding to the iteration of one of these generators which yields curves. The analysis of the polynomial growth of the degree of these iterations is sketched in Appendix E and leads to a quite simple generating function:

$$G_{\text{BTA}}(x) = \frac{1+2x^3}{(1-x)^3\,(1+x)} \tag{6.16}$$

This greater complexity of the generating function one encounters with edge-spin models comes from the fact that the involution which plays the role of the transpositions t_1, t_2,... for vertex models is a nonlinear transformation (namely the Hadamard inverse[5,6]) which amounts to taking the inverse of each entry of the matrix $R[i, j] \to 1/R[i, j]$. One cannot find, as simply as for vertex models, "Plücker-like" variables[1] of a reasonable

[18] The birational transformations corresponding to these two examples of spin-edge models[6,37] can be "q-deformed," this deformation preserving the integrability (namely the foliation in elliptic curves of the parameter space).[14,37] It is worth noticing that these q-deformed birational transformations have the same generating functions as (6.12) or (6.14).

degree that "linearize" the action of the matrix inversion I and of the other involution: the algebraic expressions covariant under the action of the action of the matrix inversion I and of the Hadamard inverse are of a higher degree.[5,6] In fact, it is always possible, after Kadanoff and Wegner,[39] to map a spin-edge model for which the edge Boltzmann weight interaction depends on the *difference* between nearest-neighbor spins onto a vertex model.[39] Introducing the edge Boltzmann weight interaction $W(\sigma_i - \sigma_j)$ (associated with the horizontal bonds) between two neighboring spins σ_i and σ_j, and $\bar{W}(\sigma_k, \sigma_l)$ another edge Boltzmann weight (associated with the vertical bonds between two neighboring spins σ_k and σ_l), the two bonds $[\sigma_i - \sigma_j][\sigma_k - \sigma_l]$ being dual bonds, one can easily associate a vertex Boltzmann weight given by

$$W_{\text{vert}}(i, j, k, l) = W(\sigma_i - \sigma_j) \cdot \bar{W}(\sigma_k - \sigma_l) \tag{6.17}$$

with $i = \sigma_i - \sigma_k$, $j = \sigma_k - \sigma_j$, $k = \sigma_j - \sigma_l$, and $l = \sigma_l - \sigma_i$, and therefore $i + j + k + l = 0$.

This transformation maps the edge-spin model onto a vertex model, thus allowing one to introduce linear involutions like $t_1, t_2, \ldots$. However, this "linearization" of the problem multiplies by two the degree of all the algebraic expressions encountered.

7. CONCLUSION

We have used the methods introduced in refs. 1–3 on various examples of vertex models of lattice statistical mechanics. In particular, we have analyzed the factorization properties of discrete symmetries of the parameter space of these lattice models, represented as birational transformations.

For all the examples introduced in this paper, which correspond to matrices of arbitrary size, it has been shown that remarkable factorization relations *independent of the matrix size* occur [see, for instance, (3.14), (5.6)].

Different features have eerged from this study, namely the *polynomial growth of the complexity* of the iterations of these birational transformations, the existence of recursion relations bearing on the factorized polynomials f_n, and the existence of deterinantal compatibility conditions like (5.10). The relation between these properties and the integrability of these lattice models of statistical mechanics, or more general structures like the "quasiintegrability,"[10] has been studied. The analysis of the factorizations corresponding to a specific two-dimensional vertex model (the Stroganov model; see Sections 3.3 and 3.4) has shown how the generic exponential growth of the calculations does reduce to a polynomial growth when the model becomes Yang–Baxter integrable. This gives a first example of the fact that the search for *polynomial growth* of the complexity of the associated iterations provides a new way to analyze *vertex models.*[8,9,31]

It has been shown that the (determinantal) compatibility condition associated with the "pre-Bethe Ansatz" (5.10) naturally yields *algebraic curves* of quite high genus *together with their associated Jacobian variety*: one could seek systematically for models (that is, specific patterns of matrix Boltzmann weight) for which this genus becomes as small as possible. These compatibility conditions yield as many curves (5.10) as the dimension d of the lattice (see Remark 1 in Appendix C): one should concentrate on the models for which the d algebraic curves (5.10) are, as much as possible, on the same footing (same genus,...).

The examples of birational transformations associated with *vertex* models, detailed here, enable us to clarify the occurrence of polynomial growth of the complexity of the iterations: in particular, it has been shown, using the examples of three-dimensional vertex models, that a polynomial growth not only occurs with algebraic elliptic *curves*,[3,12] but can also occur for transformations yielding *algebraic surfaces or even higher-dimensional varieties.* In this respect a very general three-dimensional vertex model, the 64-state vertex model, emerges as a remarkable model illustrating such a situation (see Section 4).

In fact, it will be shown in forthcoming publications that one can prove the polynomial growth of the calculations[36] when the transformation K can be represented as a *shift* on a Jacobian variety C^n/Γ.

The search for *polynomial growth* of the complexity of the associated iterations could provide a new way to analyze *three- or higher-dimensional vertex models,*[8,9,31] searching systematically for models where a Jacobian variety of an algebraic curve occurs.[19]

Let us recall that Jacobian varieties of curves are particular Abelian varieties depending only on $3g-3$ moduli among the $g(g+1)/2$ parameters[20] upon which the Abelian varieties depend.

Conversely, it is not clear whether a polynomial growth necessarily implies the existence of an associated *Jacobian variety* (one can imagine a situation where Abelian varieties which are not Jacobian varieties occur together with polynomial growth, or $K3$ surfaces together with polynomial growth,...). We will try in further publications to see if this polynomial growth is necessarily related to *Abelian varieties.* We will also try to see to what extent the *product* of elliptic curves is a situation favored in lattice statistical mechanics.

[19] Particular attention may be devoted to the subcase of *hyperelliptic curves*: the (analytical) $(3g-3)$-dimensional space of moduli (Teichmüller space) has singularities corresponding to the hyperelliptic curves which only depend on $2g-1$ moduli.

[20] The period matrix of the theta functions of g variables has to be symmetric.

APPENDIX A. POLYNOMIALS ASSOCIATED WITH STROGANOV'S MODEL

The X_n^H, Y_n and Z_n defined in Section 3.4 read, up to $n=5$,

$$X_1^H = b-1, \qquad Y_1 = 2b+1, \qquad Z_1 = b+1$$

$$X_2^H = 2b^2 - b - 1 + bc + c, \qquad Y_2 = 4b^2 - 2b - 2 - bc - c$$

$$Z_2 = 2b^2 - b - 1 - bc - c$$

$$X_3^H = 4b^3 + 2b^2c - 6b^2 - bc^2 - bc + 2 - c^2 - c$$

$$\begin{aligned} Y_3 = {} & 4 - 3bc - 12b^2 + 8b - 2c + 2b^3c + 3b^2c - b^2c^2 \\ & - 3bc^2 - 2c^2 + 16b^4 - 16b^3 \end{aligned}$$

$$Z_3 = 2 - bc - 6b^2 + 4b - c + 2b^2c - bc^2 - c^2 + 8b^4 - 8b^3$$

$$\begin{aligned} X_4^H = {} & 16b^5 - 32b^4 + 16b^4c + 4b^3 - 12b^3c - 18b^2c + 20b^2 \\ & - b^2c^3 + b^2c^2 - 3bc^3 - bc^2 - 4b + 8bc - 4 + 6c - 2c^3 \end{aligned}$$

$$\begin{aligned} Y_4 = {} & 16 - 34bc - 96b^2 + 32b - 16c + 106b^3c + 62b^2c + 20b^2c^2 - 27bc^2 \\ & - 12c^2 + 272b^4 - 128b^3 + 8c^3 + 96b^5 - 320b^6 - 110b^4c + 57b^3c^2 \\ & + 19bc^3 + 72b^6c - 32b^5c^2 - 80b^5c - 8b^4c^3 - 6b^4c^2 \\ & - 19b^3c^3 + 2b^3c^4 + 8b^2c^4 + 10bc^4 + 4c^4 + 128b^7 \end{aligned}$$

$$\begin{aligned} Z_4 = {} & 8 - 16bc - 48b^2 + 16b - 8c + 48b^3c + 32b^2c + 10b^2c^2 - 14bc^2 + 136b^4 \\ & - 64b^3 + 4c^3 + 48b^5 - 160b^6 - 56b^4c + 30b^3c^2 + 9bc^3 + 32b^6c - 16b^5c^2 \\ & - 32b^5c - 4b^4c^3 - 4b^4c^2 - 9b^3c^3 + b^3c^4 + 4b^2c^4 + 5bc^4 + 2c^4 + 64b^7 \end{aligned}$$

$$\begin{aligned} X_5^H = {} & 48bc - 16 + 128b^2 + 10bc^5 - 16b + 32c + 8b^2c^5 - 40b^5c^3 \\ & - 5b^4c^4 + 192b^7c + 32b^6c^2 + 4c^5 + 128b^8 - 160b^3c - 192b^2c^2 + 16b^2c^2 \\ & - 16bc^2 - 4c^2 - 400b^4 + 32b^3 - 20c^3 + 176b^5 + 416b^6 + 480b^4c \\ & + 56b^3c^2 + 39b^2c^3 - 38bc^3 - 448b^6c - 40b^5c^2 + 48b^5c \\ & - 17b^4c^3 - 44b^4c^2 + 76b^3c^3 - 12b^3c^4 + b^2c^4 + 12bc^4 + 4c^4 - 448b^7 \end{aligned}$$

$$\begin{aligned} Y_5 = {} & -137c^6b^2 - 130c^6b - 32c^6 + 24576b^{10} - 128 + 1032bc + 1152b^2 \\ & + 350bc^5 - 384b + 320c + 200b^2c^5 - 562b^3c^5 + 1662b^5c^4 - 6886b^5c^3 \\ & - 794b^4c^4 + 32448b^8c - 26424b^7c - 782b^6c^3 - 696b^4c^5 \\ & + 5968b^7c^2 + 1888b^6c^2 + 96c^5 + 5120b^9 - 34560b^8 - 7368b^3c \end{aligned}$$

$$
\begin{aligned}
&-2096b^2c+352b^2c^2-368bc^2-96c^2-5760b^4+3200b^3-272c^3\\
&-10368b^5+18816b^6+7776b^4c+2224b^3c^2+602b^2c^3-926bc^3\\
&-20784b^6c-5456b^5c^2+21048b^5c+28b^4c^3-704b^4c^2+4084b^3c^3\\
&-1546b^3c^4+27b^2c^4+490bc^4+128c^4+12672b^7+7744b^9c\\
&+5200b^7c^3-19968b^{10}c+488b^8c^3-3104b^8c^2+959b^6c^4+82b^5c^5\\
&+44b^3c^6+6272b^{11}c+1152b^{10}c^2-1856b^9c^2-312b^8c^4\\
&-18432b^{11}-16c^7+4096b^{12}-614b^7c^4-1536b^9c^3\\
&+130b^7c^5+400b^6c^5+15b^6c^6+86b^5c^6+154b^4c^6\\
&-4b^5c^7-28b^4c^7-76b^3c^7-100b^2c^7-64bc^7\\
Z_5=\ &-66c^6b^2-64c^6b-16c^6+12288b^{10}-64+512bc+576b^2+176bc^5\\
&-192b+160c+102b^2c^5-388b^6c^3-352b^4c^5-282b^3c^5+817b^5c^4\\
&-3456b^5c^3-361b^4c^4+16128b^8c-12864b^7c+2688b^7c^2+1056b^6c^2\\
&+48c^5+2560b^9-17280b^8-14b^4c^7-38b^3c^7-50b^2c^7-32bc^7\\
&-3648b^3c-1056b^2c+192b^2c^2-176bc^2-48c^2-2880b^4+1600b^3\\
&-136c^3-5184b^5+9408b^6+3936b^4c+1056b^3c^2+300b^2c^3\\
&-464bc^3-10464b^6c-2544b^5c^2+10368b^5c+16b^4c^3-432b^4c^2\\
&+2048b^3c^3-759b^3c^4+2b^2c^4+240bc^4+64c^4+6336b^7+3584b^9c\\
&+2608b^7c^3-9728b^{10}c+240b^8c^3-1536b^8c^2+443b^6c^4+40b^5c^5\\
&+23b^3c^6+3072b^{11}c+512b^{10}c^2-768b^9c^2-144b^8c^4-9216b^{11}\\
&-8c^7+2048b^{12}-302b^7c^4-768b^9c^3+66b^7c^5\\
&+202b^6c^5+7b^6c^6+41b^5c^6+75b^4c^6-2b^5c^7
\end{aligned}
$$

APPENDIX B

B1. Restricted Factorizations in Dimension Three: Product of Elliptic Curves

Similar to what is done in Section 3.3, one can consider the "restricted factorization problem" corresponding to the following initial matrix:

$$
R_{3D}=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{B.1}
$$

where the 4×4 submatrices A, B, C, and D are of the form

$$A = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix} \tag{B.2}$$

where A_1, A_2, and A_3 are 2×2 matrices and 0 denotes the 2×2 matrix with zero entries, and the form for matrices B, C, and D is similar to (B.2). It is straightforward to see that a form like (B.1), together with (B.2), is actually compatible with the action of the group generated by the matrix inverse $\hat{I}$ and t_1.[7,8]

For such matrices (B.1) and (B.2) one can see (permuting rows and columns 3–4 and 5–6 of the 8×8 matrix R_{3D}) that the polynomials f_n defined by Eqs. (4.2) factorize into the product of two polynomials. One can show that these two polynomials $F_n^{(1)}$ and $F_n^{(3)}$ actually correspond to the action the birational transformation K associated with two 16-vertex models (see Section 3.1) associated with the following two 4×4 matrices:

$$M_0^{(1)} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \text{and} \quad M_0^{(3)} = \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix} \tag{B.3}$$

One gets therefore for the f_n

$$f_n = F_n^{(1)} \cdot F_n^{(3)} \tag{B.4}$$

where each of the $F_n^{(1)}$ and $F_n^{(3)}$ satisfy *independently* the *same* recursion relation, which is actually the recurrence occurring for the 16-vertex model (see Section 3.1). This nonlinear recursion relation has been shown to yield *algebraic elliptic curves* $\mathscr{E}$.[1-3] It is thus clear, at least in subcase (B.2), that the f_n do not satisfy a recursion relation [like (3.12)], but that the orbits of the iteration of K are naturally associated with *algebraic surfaces* which are the *product of two algebraic elliptic curves*: $\mathscr{S} = \mathscr{E} \times \mathscr{E}$.

From Eq. (B.4), one easily gets in this subcase [(B.1), (B.2)] that the degrees of the f_n and $\det(M_n)$, namely β_n and α_n, can be written as sums of two terms:

$$\beta_n = \beta_n^{(1)} + \beta_n^{(3)}, \qquad \alpha_n = \alpha_n^{(1)} + \alpha_n^{(3)} \tag{B.5}$$

where the $\beta_n^{(i)}$ (resp. the $\alpha_n^{(i)}$) are the degrees of the $F_n^{(i)}$ [resp. the $\det(M_n)^{(i)}$] with $i = 1, 3$.

From Section 3.1 [see Eqs. (3.7)], one immediately gets that $\beta_n^{(1)} = \beta_n^{(3)} = 2n(n+1)$ and $\alpha_n^{(1)} = \alpha_n^{(3)} = 4(2n^2+1)$. This provides an example of a *quadratic growth associated with an algebraic surface* (namely: $\mathscr{S} = \mathscr{E} \times \mathscr{E}$).

B2. A Three-Dimensional Generalization of the Six-Vertex Model

Another example of "restricted factorization" corresponds to the vertex model defined in refs. 8, 9, and 31, which can be seen as a three-dimensional generalization of the *six-vertex model*.[31] It corresponds to the K-compatible conditions (4.1) together with the additional conditions

$$R^{i_1 i_2 i_3}_{+1+1+1} = 0 \qquad \text{if} \quad (i_1, i_2, i_3) \neq (+1, +1, +1)$$

and

$$R^{i_1 i_2 i_3}_{-1-1-1} = 0 \qquad \text{if} \quad (i_1, i_2, i_3) \neq (-1, -1, -1)$$

and

$$R^{+1,+1,+1}_{j_1 j_2 j_3} = 0 \qquad \text{if} \quad (j_1, j_2, j_3) \neq (+1, +1, +1) \tag{B.6}$$

and

$$R^{-1,-1,-1}_{j_1 j_2 j_3} = 0 \qquad \text{if} \quad (j_1, j_2, j_3) \neq (-1, -1, -1)$$

This particular form for the 8×8 matrix (B.6) is not stable by the transformation K [basically because the transposition t_1 does not preserve the form (B.6)], but it is *preserved under the action of* K^2.[(31),21] Taking into account the simplicity of this model, one can relax the matrix symmetry condition (4.22). This gives [with notations (4.13)] the following 8×8 matrix:

$$R^{3d} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & c & 0 & n & 0 & 0 & 0 \\ 0 & e & f & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & i & j & 0 \\ 0 & j & i & 0 & h & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & f & e & 0 \\ 0 & 0 & 0 & n & 0 & c & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \tag{B.7}$$

or, recalling the relabeling previously introduced $\{(1, 2, 3, 4, 5, 6, 7, 8) \rightarrow (1, 4, 6, 7, 8, 5, 3, 2) = [(+1, +1, +1), (+1, -1, -1), (-1, +1, -1), (-1, -1, +1)]$

[21] This situation generalizes the one encountered in two dimensions with six-vertex models.

and $[(-1, -1, -1), (-1, +1, +1), (+1, -1, +1), (+1, +1, -1)]\}$, we can write the two identical 4×4 matrices (4.14)

$$B^{3d}=\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & h & i & j \\ 0 & p & f & e \\ 0 & n & c & b \end{pmatrix} \tag{B.8}$$

This model depends on *ten homogeneous parameters*. For this three-dimensional generalization of the six-vertex model[9,31] one can introduce the same f_n as the ones given in Section 4.2. The corresponding matrices can be seen to be products of 3×3 and 1×1 matrices. One straightforward consequence is that all the determinants one calculates are perfect squares which factorize into determinants of 3×3 matrices and terms corresponding to the 1×1 blocks. This enables us to introduce variables which are the determinants of these 3×3 matrices, which will be denoted g_n in the following, instead of the variables f_n related to the determinant of the whole 8×8 matrix. Introducing two variables w_0 and w_1 related to two particular entries of the matrix M_0 and its transform by K, one gets factorizations

$$\begin{gathered} w_0=(M_0)_{11}, \qquad g_1=\frac{(\det(M_0))^{1/2}}{w_0}, \qquad M_1=\frac{K(M_0)}{g_1 w_0} \\ w_1=\frac{(M_1)_{44}}{w_0}, \qquad g_2=\frac{(\det(M_1))^{1/2}}{w_0^3 w_1}, \qquad M_2=\frac{K(M_1)}{g_2 w_0^5 w_1} \\ g_3=\frac{(\det(M_2))^{1/2}}{w_0 w_1^3 g_1^3}, \qquad M_3=\frac{K(M_2)}{w_0 w_1^5 g_1^5 g_3} \\ g_4=\frac{(\det(M_3))^{1/2}}{w_0^3 w_1 g_2^3}, \qquad M_4=\frac{K(M_3)}{w_0^5 w_1 g_2^5 g_4},\ldots \end{gathered} \tag{B.9}$$

and, for arbitrary n,

$$\begin{aligned} K(M_n) &= M_{n+1}\cdot w_0\cdot w_1^5\cdot g_{n-1}^5\cdot g_{n+1} \qquad \text{for} \quad n \text{ even} \\ K(M_n) &= M_{n+1}\cdot w_0^5\cdot w_1\cdot g_{n-1}^5\cdot g_{n+1} \qquad \text{for} \quad n \text{ odd} \end{aligned} \tag{B.10}$$

together with

$$\begin{aligned} (\det(M_n))^{1/2} &= g_{n+1}\cdot w_0\cdot w_1^3\cdot g_{n-1}^3 \qquad \text{for} \quad n \text{ even} \\ (\det(M_n))^{1/2} &= g_{n+1}\cdot w_0^3\cdot w_1\cdot g_{n-1}^3 \qquad \text{for} \quad n \text{ odd} \end{aligned} \tag{B.11}$$

yielding for $n \geqslant 2$

$$\hat{K}(M_n) = \frac{K(M_n)}{\det(M_n)} = \frac{M_{n+1}}{g_{n-1} \cdot g_{n+1} \cdot w_0 \cdot w_1} \tag{B.12}$$

Similarly, one can introduce the degrees of the determinants of these matrices M_n and the degrees of the successive polynomials g_n, namely α_n and $\hat{\beta}_n$ and their corresponding generating functions $\alpha(x)$ and $\hat{\beta}(x)$. These generating functions read

$$\alpha(x) = \frac{8(1+x+x^2-x^3)}{(1+x)(1-x)^3}, \qquad \hat{\beta}(x) = \frac{x(3-2x)}{(1-x)^3} \tag{B.13}$$

The exponents α_n and $\hat{\beta}_n$ read

$$\alpha_n = 4n^2 + 16n + 6 + 2(-1)^n, \qquad \hat{\beta}_n = \frac{n^2+5n}{2} \tag{B.14}$$

Again, one can study the "right action" of K on matrices M_n [Eqs. (3.3)]. However, since the form of mtrix (B.7) is only preserved by K^2, the right action is a little bit more involved, namely

$$\begin{gathered} 2(g_1)_{K^2} = g_3\, g_2^3\, g_1^{23}\, w_0^6 w_1^5, \qquad 8(g_2)_{K^2} = g_4\, g_2^9\, g_1^{53}\, w_0^{84}\, w_1^{11} \\ 64(g_3)_{K^2} = g_5\, g_2^{17} g_1^{90}\, w_0^{144}\, w_1^{18} \cdots \end{gathered} \tag{B.15}$$

and for arbitrary n

$$2^{z_n^{(1)}}(g_n)_{K^2} = g_{n+2}\, g_2^{z_n^{(2)}}\, g_1^{z_n^{(3)}}\, w_0^{z_n^{(4)}}\, w_1^{z_n^{(5)}} \tag{B.16}$$

where the $z_n^{(i)}$ are quadratic integers:

$$z_n^{(1)} = \frac{n(n+1)}{2}, \qquad z_n^{(2)} = n^2 + 3n - 1, \qquad z_n^{(3)} = \frac{(7n+39)n}{2}$$

$$z_n^{(4)} = 6n(n+5), \qquad z_n^{(5)} = \frac{(n+9)n}{2}$$

The factorization scheme (B.10), (B.11) is not modified when the matrix symmetry condition (4.22) hold. In contrast, if one relaxes the spin reversal condition (4.10) (20 homogeneous parameters), the factorization scheme (B.10) is reminiscent of the "stringlike" factorizations (4.3).

A more detailed analysis, with particular emphasis on the "pre-Bethe-ansatz" conditions [see ref. 10 and (C.2) in the following], of this three-dimensional generalization of the six-vertex model has been performed in ref. 31.

APPENDIX C. PRE-BETHE ANSATZ

Let us given here miscellaneous remarks explaining the occurrence of algebraic curves like (5.10) in the analyzis of vertex models. For this purpose let us first recall the relevance of a key "factorization" relation compatible with the action of the birational transformations K, namely the pre-Bethe-Ansatz condition.[10] Let us first recall the results of ref. 10 on the *16-vertex model* (which corresponds to $m=2$ in the previous section). The weak-graph duality[20] symmetries correspond to a "gauge group" $G = sl_2 \times sl_2$ which acts *linearly* on R by similarity transformtions (see ref. 20 for details):

$$\text{if} \quad g = g_1 \times g_2, \qquad g(R) = g_1^{-1} g_2^{-1} \cdot R \cdot g_1 g_2 \tag{C.1}$$

Let us denote by $\mathscr{B}$ the group of birational transformations generated by I, t_1, and t_2. The actions of G and $\mathscr{B}$ do not commute. However, G and I do commute, and t_1 (resp. t_2) sends orbits of G onto orbits of G. A group larger than the gauge group G has naturally emerged in the analysis of the symmetries of the 16-vertex model, a group we have denoted G_{Bethe}.[10] Actually one of the keys to the Bethe Ansatz is the existence [see Eqs. (B.10), (B.11a) in ref. 40] of vectors which are pure tensor products (of the form $v \otimes w$) and which R maps onto the pure tensor product $v' \otimes w'$ (see also refs. 10, 34, and 35). If

$$v = \begin{pmatrix} 1 \\ p \end{pmatrix}, \qquad w = \begin{pmatrix} 1 \\ q \end{pmatrix}, \qquad v' = \begin{pmatrix} 1 \\ p' \end{pmatrix}, \qquad w' = \begin{pmatrix} 1 \\ q' \end{pmatrix}$$

then the solution of the "pre-Bethe-Ansatz" equation[10]

$$R(v \otimes w) = \mu v' \otimes w' \tag{C.2}$$

satisfies the two *biquadratic relations*[10]

$$\begin{aligned} l_4 + l_{11}\, p - l_{12}\, p' + l_2\, p^2 + l_1\, p'^2 - (l_9 + l_{18})\, pp' - l_{13}\, p^2 p' \\ + l_{10}\, pp'^2 + l_3\, p^2 p'^2 = 0 \end{aligned} \tag{C.3}$$

$$\begin{aligned} l_7 + l_{16} q - l_{15} q' + l_8 q^2 + l_5 q'^2 - (l_9 - l_{18})\, qq' - l_{17} q^2 q' \\ + l_{14} qq'^2 + l_6 q^2 q'^2 = 0 \end{aligned} \tag{C.4}$$

These two biquadratics *are elliptic curves.* Remarkably, when calculating the *modular invariant*[1] of these curves, one can actually see that these two curves actually reduce to *the same Weierstrass canonical form*[10,42]

$$y^2 = 4x^3 - g_2 x - g_3 \tag{C.5}$$

A group $G_{\rm Bethe} \simeq sl_2 \times sl_2 \times sl_2$ naturally acts on (C.2): the four copies of sl_2 act respectively on v, w, v', w'. This induces a linear action on R:

$$R \to g_{1L}^{-1} \cdot g_{2L}^{-1} \cdot R \cdot g_{1R} \cdot g_{2R} \tag{C.6}$$

The infinite-order transformations K and K_{t_2} can both be represented as a *shift* of the (spectral) parameter[10] enabling one to move along these various elliptic curves: the two biquadratics (C.3) and (C.4) and the elliptic curves generated by transformations K and K_{t_2} in $\mathbb{C}P_{15}$. This situation can straightfowardly be generalized to $2m \times 2m$ matrices [see the transposition t_1 defined by (5.1) in Section 5], but now directions 1 and 2 are no longer on the same footing: vectors w and w' have m coordinates instead of two. Their elimination still yields a reltion similar to (C.3) but now of a higher degree. The linear action (C.6) is changed into

$$R \to g_{2L}^{-1} \cdot R \cdot g_{2R} \tag{C.7}$$

Let us represent g_{2R} and g_{2L}^{-1} as $2m \times 2m$ matrices, namely

$$g_{2R} = \begin{pmatrix} G_{2R} & 0 \\ 0 & G_{2R} \end{pmatrix} \quad \text{and} \quad g_{2L}^{-1} = \begin{pmatrix} G_{2L}^{-1} & 0 \\ 0 & G_{2L}^{-1} \end{pmatrix} \tag{C.8}$$

where G_{2R} and G_{2L} are two $m \times m$ matrices.

Using notations (5.1) for the Boltzmann weight matrix, one can easily see that this elimination *yields the following determinantal relation between p and p'*:

$$\det(Ap' - C - Dp + pp'B) = 0 \tag{C.9}$$

This determinant is a polynomial of degree m *in each variable* p and p'. It is important to note that this determinant is covariant under the "gaugelike" transformtions (C.8):

$$\begin{aligned} &\det(Ap' - C - Dp + pp'B) \\ &\quad \to \det(G_{2R})^2 \cdot \det(G_{2L})^{-2} \cdot \det(Ap' - C - Dp + pp'B) \end{aligned} \tag{C.10}$$

The compatibility condition (5.10) is therefore invariant under the "gaugelike" transformations (C.7).

We have performed an analysis for the $m = 4$ case (more precisely, for an 8×8 Boltzmann matrix corresponding to a three-dimensional vertex model), getting *biquartic* relations.[31] Generally, for $2m \times 2m$ matrices, one gets relations of degree m *both in p and p'*. Curve (C.9), except for the remarkable $m = 2$ case (the 16-vertex model!), for which the *curve identifies* *with its Jacobian*, is a curve of genus greater than one. Generically it is a curve of genus[22]

$$g = (2m-2)(2m-1)/2 - 2(m-1)m/2 = (m-1)^2$$

and Korepanov[(32)] and Krichever[(35)] have claimed that the group of birational transformations we study can actually be represented as a *shift* on the Jacobian variety assoiated with curve (5.10). The transformation K *linearizes on the Jacobian variety*: the transformation K corresponds to a constant shift on the torus. The transformation K amounts to adding a fixed element of the Albanese variety C^g/Γ.

Remark 1. The analysis of $2m \times 2m$ matrices can be seen as a preliminary study for the $2^d \times 2^d$ matrices corresponding to d-dimensional problems (see Section 4). One can similarly write down a d-dimensional "pre-Bethe Ansatz" condition[(10)]

$$R(v_1 \otimes v_2 \otimes \cdots \otimes c_d) = \mu v'_1 \otimes v'_2 \otimes \cdots \otimes v'_d \tag{C.11}$$

The elimination of $2(d-1)$ vectors (for instance, $v_2 \cdots v_d$ and $v'_2 \cdots v'_d$) yields d algebraic curves like (5.10). Among the various d-dimensional R-matrices, the ones for which the genus of the previous d algebraic curves are *all equal* and smaller than $(m-1)^2 = (2^{d-1}-1)^2$ are of particular interest.

Remark 2. Further Generalizations. The "pre-Bethe-Ansatz" condition (C.2) can be generalized to $(n \cdot m) \times (n \cdot m)$ matrices. Again vectors w and w' have m coordinates instead of two, but now the vectors v and v' have n components. Let us just write here the $n = 3$ case. Let us denote the components of vectors v and v' and the $(3m) \times (3m)$ Boltzmann matrix in terms of $m \times m$ matrices $A_1, \ldots, A_9$ as follows:

$$v = \begin{pmatrix} 1 \\ p_1 \\ p_2 \end{pmatrix} \qquad v' = \begin{pmatrix} 1 \\ p'_1 \\ p'_2 \end{pmatrix}, \qquad f_R = \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix}$$

[22] A formula for getting the genus is, for example, Noether's formula obtained assuming that the curve of degree d has only ordinary multiple points. Since we have only n-uple points, this yields $g = (d-1)(d-2)/2 - Nn(n-1)/2$, where N is the number of n-uple points.[(43–45)] We have here *two* n-uple points. To see this one can, for instance, write curve (5.10) in a homogeneous way, as the intersection of equations $\det(Ap' - Ct - Dp + t'B) = 0$ and $pp' = tt'$.

The elimination of vectors w and w' yields two determinantal conditions (instead of one for $n=2$ and, generally, $n=1$ conditions for arbitrary n):

$$\det((A_1 + A_2 p_1 + A_3 p_2) \cdot p_2' - (A_7 + A_8 p_1 + A_9 p_2)) = 0 \qquad \text{(C.12)}$$

$$\det((A_1 + A_2 p_1 + A_3 p_2) \cdot p_1' - (A_4 + A_5 p_1 + A_6 p_2)) = 0 \qquad \text{(C.13)}$$

Curve (5.10) is thus replaced, for $n=3$, by an *algebraic surface* given by the two conditions (C.12) and (C.13), and, for arbitrary n, by an $(n-1)$-dimensional algebraic variety given by $n-1$ "determinantal conditions" bearing on $2(n-1)$ variables $p_1, ..., p_{n-1}$ and $p_1', ..., p_{n-1}'$.

This simple remark enables us to understand better *why the number of colors two*, for the arrows of the vertex models, *plays such a special role for the occurrence of polynomial growth.*

APPENDIX D. GAUGE TRANSFORMATIONS

The two transformations g_{2R} and g_{2L}^{-1} are actually *symmetries* of the transformation K corresponding to the transposition t_1 defined by (5.1). With notations (7.24) one easily gets

$$K(g_{2L}^{-1} \cdot M \cdot g_{2R}) = \det(g_{2R}) \cdot \det(g_{2L})^{-1} \cdot g_{2R}^{-1} \cdot K(M) \cdot g_{2L} \qquad \text{(D.1)}$$

$$K^2(g_{2L}^{-1} \cdot M \cdot g_{2R}) = (\det(g_{2R}) \cdot \det(g_{2L})^{-1})^{2(m-1)} \cdot g_{2L}^{-1} \cdot K^2(M) \cdot g_{2R} \qquad \text{(D.2)}$$

and for arbitrary n

$$\begin{aligned} K^{2n}(g_{2L}^{-1} \cdot M \cdot g_{2R}) &= (\det(g_{2R}) \cdot \det(g_{2L})^{-1})^{z_{2n}} \cdot g_{2L}^{-1} \cdot K^{2n}(M) \cdot g_{2R} \\ &= (\det(G_{2R}) \cdot \det(G_{2L})^{-1})^{2z_{2n}} \cdot g_{2L}^{-1} \cdot K^{2n}(M) \cdot g_{2R} \end{aligned} \qquad \text{(D.3)}$$

$$\begin{aligned} K^{2n+1}(g_{2L}^{-1} \cdot M \cdot g_{2R}) &= (\det(g_{2R}) \cdot \det(g_{2L})^{-1})^{z_{2n+1}} \cdot g_{2R}^{-1} \cdot K^{2n+1}(M) \cdot g_{2L} \\ &= (\det(G_{2R}) \cdot \det(G_{2L})^{-1})^{2z_{2n+1}} \cdot g_{2R}^{-1} \cdot K^{2n+1}(M) \cdot g_{2L} \end{aligned} \qquad \text{(D.4)}$$

with

$$z_{2n} = \frac{(2m-1)^{2n} - 1}{2m}, \qquad z_{2n+1} = \frac{(2m-1)^{2n+1} + 1}{2m} \qquad \text{(D.5)}$$

For the inhomogeneous transformations $\hat{K}$ one also gets

$$\begin{aligned} \hat{K}^{2n}(g_{2L}^{-1} \cdot M \cdot g_{2R}) &= g_{2L}^{-1} \cdot \hat{K}^{2n}(M) \cdot g_{2R} \\ \hat{K}^{2n+1}(g_{2L}^{-1} \cdot M \cdot g_{2R}) &= g_{2R}^{-1} \cdot \hat{K}^{2n+1}(M) \cdot g_{2L} \end{aligned} \qquad \text{(D.6)}$$

This is a simple consequence of the relations corresponding to the two transformations I and t_1:

$$\begin{aligned} I(g_{2L}^{-1} \cdot M \cdot g_{2R}) &= \det(g_{2R}) \cdot \det(g_{2L})^{-1} \cdot g_{2R}^{-1} \cdot I(M) \cdot g_{2L} \\ &= \det(G_{2R})^2 \cdot \det(G_{2L})^{-2} \cdot g_{2R}^{-1} \cdot I(M) \cdot g_{2L} \end{aligned}$$

and

$$t_1(g_{2L}^{-1} \cdot M \cdot g_{2R}) = g_{2L}^{-1} \cdot t_1(M) \cdot g_{2R} \tag{D.7}$$

If one imposes $\det(G_{2R}) = \det(G_{2L})$, one gets an invariance under the homogeneous transformations K^{2n}, but in fact one has, in general, a *covariance* property which *is actually a symmetry* closely linked to the homogeneity of the problem.[1,2] The f_n and $\det(M_n)$ transform very simply under (C.7):

$$\begin{aligned} f_n &\to (\det(g_{2R}) \cdot \det(g_{2L})^{-1})^{\beta_n/2m} \cdot f_n \\ &= (\det(g_{2R}) \cdot \det(g_{2L})^{-1})^{n(n+1)/2} f_n \end{aligned} \tag{D.8}$$

$$\det(M_n) \to (\det(g_{2R}) \cdot \det(g_{2L})^{-1})^{\alpha_n/2m} \cdot \det(M_n) \tag{D.9}$$

In the $m = 2$ case (16-vertex model; see Section 3.1) the f_n satisfy recursion relations like (3.12), yielding elliptic curves. These relations are actually *invariant* under symmetry (D.8) (see, for instance, refs. 1 and 2). One notes from (D.6) that *the inhomogeneous variables* $x_n = l_n \cdot l_{n+1}$, the product of two consecutive l_n $[l_n = \det(\hat{K}^n(M_0))]$ [and therefore recursions (3.12)] *are actually invariant under* (C.7): variables x_n actually "gauge-away" this quite large symmetry group (C.7). When considering the iterations of K^2 or $\hat{K}^2$, one can, without any loss of generality, "gauge-away" the parameters corresponding to these (linear) transformations (C.8): one has two times $m^2 - 1$ inhomogeneous parameters corresponding to G_{2L} and G_{2R}.

APPENDIX E. SOME GENERATING FUNCTIONS FOR SPIN MODELS

Let us sketch here the analysis of the growth of the complexity of the iterations for the two- and three-site interaction q-state standard scalar Potts model on the triangular lattice.[17,38,46] One can introduce a $q \times q$ matrix Boltzmann weight for this model.[38,46] One inversion relation, the transformation I_h, is the (homogeneous) matrix inversion, while other symmetries, playing the role of transformations t_i in this paper, are permutations of the entries of this $q \times q$ Boltzmann matrix.[38] Let us consider an infinite-order (homogeneous) birational transformation, which we denote

K, obtained from I_h and one of these permutations (this transformation is the transformation $p_{12}I_h$ in ref. 38). Similar to the situation encountered with the three-dimensional generalization of the six-vertex model (see Section 4.2.3), the determinant is replaced by two of its factors, which we denote P_1 and P_2. One has the following factorizations:

$$K(M_{n+4}) = c_n c_{n+1} d_{n+1} \cdot M_{n+5} \tag{E.1}$$

where the c_n and d_n are (homogeneous) factorizing polynomials, and

$$P_1(M_n) = c_n c_{n-3} d_{n-3} \tag{E.2}$$

$$P_2(M_n) = c_{n-1} c_{n-3} c_{n-4}^2 d_n d_{n-3} \tag{E.3}$$

where P_1 and P_2 are polynomials, respectively of degree 1 and 2 in the entries of the matrix M_n. These two polynomials are in fact the two prime factors of the determinant of the matrix M_n.

Introducing polynomials f_n such that (E.1) reads

$$K(M_n) = f_n \cdot M_{n+1} \tag{E.4}$$

that is, $f_{n+4} = c_n c_{n+1} d_{n+1}$, we have that the product of $P_1(M_n)$ and $P_2(M_n)$ reads

$$P_1(M_n)\, P_2(M_n) = f_n^2 \cdot f_{n+3} \tag{E.5}$$

Introducing α_n, the degree of the *entries*[23] of the M_n, and β_n, the degree of the f_n one gets from (E.4) and (E.5):

$$2\alpha_n = \beta_n + \alpha_{n+1} \tag{E.6}$$

$$3\alpha_n = 2\beta_n + \beta_{n+3} \tag{E.7}$$

The elimination of the β_n yields the recursion relation

$$\alpha_{n+4} - 2\alpha_{n+3} + 2\alpha_{n+1} - \alpha_n = 0 \tag{E.8}$$

or equivalently for the corresponding generating function $\alpha(x)$, the relation

$$\alpha(x) \cdot (1 - 2x + 2x^3 - x^4) = \alpha(x) \cdot (1-x)^3 \cdot (1+x) = P(x) \tag{E.9}$$

where $P(x)$ is a polynomial of degree 3. The first coefficients of $\alpha(x)$ read

$$\alpha(x) = 1 + 2x + 4x^2 + 8x^3 + \cdots \tag{E.10}$$

[23] Instead of the degree of the determinant of the M_n in most of this paper.

From these first coefficients one gets the expression for $P(x)$:

$$P(x) = 1 + 2x^3 \tag{E.11}$$

Relations (E.11) and (E.9) yield exact expressions for the generating function $\alpha(x) = G_{\mathrm{BTA}}(x)$ [see (6.16)].

ACKNOWLEDGMENTS

We thank M. Bellon and C. M. Viallet for many discussions and much encouragement.

This work was supported by CNRS.

REFERENCES

1. S. Boukraa, J. M. Maillard, and G. Rollet, Determinantal identities on integrable mappings, *Int. J. Mod. Phys. B* **8**:2157–2201 (1994).
2. S. Boukraa, J.-M. Maillard, and G. Rollet, Integrable mappings and polynomial growth, *Physica A* **208**:115–175 (1994).
3. S. Boukraa, J.-M. Maillard, and G. Rollet, Almost integrable mappings, *Int. J. Mod. Phys. B* **8**:137–174 (1994).
4. J. M. Maillard, Exactly solvable models in statistical mechanics and automorphisms of algebraic varieties, *Progress in Statistical Mechanics*, Vol. 3, C.-K. Hu, ed. (World Scientific, Singapore, 1988).
5. M. P. Bellon, J.-M. Maillard, and C.-M. Viallet, Integrable Coxeter groups, *Phys. Lett. A* **159**:221–232 (1991).
6. M. P. Bellon, J.-M. Maillard, and C.-M. Viallet, Higher dimensional mappings, *Phys. Lett. A* **159**:233–244 (1991).
7. M. P. Bellon, J.-M. Maillard, and C.-M. Viallet, Infinite discrete symmetry group for the Yang–Baxter equations: Spin models, *Phys. Lett. A* **157**:343–353 (1991).
8. M. P. Bellon, J.-M. Maillard, and C.-M. Viallet, Infinite discrete symmetry group for the Yang–Baxter equations: Vertex models, *Phys. Lett. B* **260**:87–100 (1991).
9. M. P. Bellon, J.-M. Maillard, and C.-M. Viallet, Rational mappings, arborescent iterations, and the symmetries of integrability, *Phys. Rev. Lett.* **67**:1373–1376 (1991).
10. M. P. Bellon, J.-M. Maillard, and C.-M. Viallet, Quasi integrability of the sixteen-vertex model, *Phys. Lett. B* **281**:315–319 (1992).
11. M. P. Bellon, J.-M. Maillard, and C.-M. Viallet, Dynamical systems from quantum integrability, in *Yang–Baxter Equations in Paris*, J.-M. Maillard, ed. (World Scientific, Singapore, 1993), pp. 95–124.
12. C. M. Viallet and G. Falqui, Singularity, complexity, and quasi-integrability of rational mappings, *Commun. Math. Phys.* **154**:111–125 (1993).
13. A. P. Veselov, Growth and integrability in the dynamics of mappings, *Commun. Math. Phys.* **145**:181–193 (1992).
14. C. M. Viallet, Baxterization, Dynamical systems and the symmetries of integrability, in *Proceedings of the Third Baltic Rim Student Seminar Helsinki 1993* (Springer, Berlin, 1994).
15. W. T. Tutte, *Studies in Graph Theory* (Mathematical Association of America, 175), Part II.

16. J. M. Maillard and R. Rammal, Some analytical consequences of the inverse relation for the Potts model, *J. Phys. A* **16**:353 (1983).
17. R. J. Baxter, H. N. V. Temperley, and S. E. Ashley, Triangular Potts model at its transition temperature, and related models, *Proc. R. Soc. Lond. A* **358**:535–559 (1978).
18. C. L. Schultz, Solvable q-state models in lattice statistics and quantum field theory, *Phys. Rev. Lett.* **46**:629–632 (1981).
19. Y. G. Stroganov, A new calculation method for partition functions in some lattice models, *Phys. Lett. A* **74**:116 (1979).
20. A. Gaaf and J. Hijmans, Symmetry relations in the sixteen-vertex model, *Physica A* **80**:149–171 (1975).
21. M. T. Jaekel and J. M. Maillard, Large q expansion on the anisotropic Potts model, *J. Phys. A* **16**:175 (183).
22. R. Rammal and J. M. Maillard, q-State Potts model on the checkerboard lattice, *J. Phys. A* **16**:1073–1081 (1983).
23. R. J. Baxter, The inversion relation method for some two-dimensional exactly solved models in lattice statistics, *J. Stat. Phys.* **28**:1–41 (1982).
24. M. T. Jaekel and J. M. Maillard, Inversion relations and disorder solutions on Potts models, *J. Phys. A* **17**:2079–2094 (1984).
25. M. T. Jaekel and J. M. Maillard, A criterion for disorder solutions of spin models, *J. Phys. A* **18**:1229 (1985).
26. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1981).
27. J. M. Maillard, The inversion relation: Some simple examples (I) and (II), *J. Phys.* (Paris) **46**:329 (1984).
28. J. M. Maillard, Automorphism of algebraic varieties and Yang–Baxter equations, *J. Math. Phys.* **27**:2776 (1986).
29. A. B. Zamolodchikov and A. B. Zamolodchikov, *Ann. Phys.* **120**:253 (1979).
30. M. P. Bellon, Addition on curves and complexity of mappings, in preparation.
31. M. P. Bellon, S. Boukraa, J.-M. Maillard, and C.-M. Viallet, Towards three-dimensional Bethe Ansatz, *Phys. Lett. B* **314**:79–88 (1993).
32. I. G. Korepanov, Vacuum curves, clssical integrable systems in discrete space-time and statistical physics, *Commun. Math. Phys.* (1994).
33. L. D. Faddeev, Integrable models in $1+1$ dimensional quantum field theory, in *Les Houches Lectures (1982)* (Elsevier, Amsterdam, 1984).
34. M. T. Jaekel and J. M. Maillard, Algebraic invariants in soluble models, *J. Phys. A* **16**:3105 (1983).
35. I. Krichever, Baxter's equations and algebraic geometry, *Funct. Anal. Pril.* **15**.
36. M. P. Bellon, Addition on curves and complexity of quasi-integrable rational mappings, in preparation.
37. M. P. Bellon, J.-M. Maillard, G. Rollet, and C.-M. Viallet, Deformations of dynamics associated to the chiral Potts model, *Int. J. Mod. Phys. B* **6**:3575–3584 (1992).
38. J. M. Maillard and G. Rollet, Hyperbolic Coxeter groups for triangular Potts models, preprint to be published in *J. Phys. A* (1994).
39. L. P. Kadanoff and F. J. Wegner, *Phys. Rev. B* **4**:3989–3993 (1971).
40. R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. II. Equivalence to a generalized ice-type lattice model, *Ann. Phys.* **76**:25–47 (1973).
41. S. Boukraa and J.-M. Maillard, Symmetries of lattice models in statistical mechanics and effective algebraic geometry, *Phys.* (Paris) *I* **3**:239–258 (1993).

42. A. W. Knapp, *Elliptic Curves* (Princeton University Press, Princeton, New Jersey, 1992).
43. R. Hartshorne, *Algebraic Geometry* (Springer-Verlag, Berlin, 1983), Appendix B, p. 438.
44. P. Griffiths and J. Harris, *Principles of Algebraic Geometry* (Wiley, New York, 1978).
45. I. R. Shafarevich, *Basic Algebraic Geometry* (Springer, Berlin, 1977), p. 216.
46. J.-M. Maillard, G. Rollet, and F. Y. Wu, Inversion relations and symmetry groups for Potts models on the triangular lattice, *J. Phys. A* **27**:3373–3379 (1994).